Structures of Diversity

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Abstract

In this thesis we consider, within a unified framework, several problems inspired by generalizations of the concept of graph capacity, introduced by Shannon in 1956 in the context of error–free communication over a noisy channel. Along the lines of the various generalizations of capacity we introduce and study diversity relations between combinatorial structures like strings over a discrete (finite or infinite) alphabet or strings of vertices of graphs (or hypergraphs). We consider a binary relation over a set of combinatorial objects a diversity relation if it is irreflexive (meaning that no object is in this relation with itself) and local, i.e. for two objects having projections (induced substructures) in this relation, the objects themselves are in this relation. Moreover, we consider and analyze forbiddance relations, which arise when a diversity relation should not hold between the pairs of objects. In case of both types of these relations we study the largest cardinality of sets of strings whose any pair (k–tuple) of elements is in the same relation.

Contents

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Notations

$[n]$	$\{1, \ldots, n\}$
$\log t$	the logarithm to the base 2 of t
$\binom{A}{k}$	$\{B : B \subseteq A, \lvert B\rvert = k\}$
$\mathbf{x}, \mathbf{y}, \mathbf{z}$	sequences of strings
x_i	the ith element of string $\mathbf{x}$
$w(\mathbf{x})$	the weight of the binary string $\mathbf{x}$, $w(\mathbf{x}) = \sum_{i=1}^{n} x_i$
$\mathbf{x}\rvert_I$	the projection of the string $\mathbf{x}$ onto the set of coordinates I
$\exp_2 k$	2^k

Introduction

Each individual is absolutely unique, yet we are not always able to distinguish them. Different people may appear indistinguishable to us. As someone once said: the only way to distinguish between two absolutely equal twins is to put them in different places! The point here is that while there is a difference between them, but not that much. Thus as a matter of fact almost always we are only interested in that particular kind of difference that creates distinguishability, or in other words in difference that makes difference. In this thesis we analyze various criteria of difference in the context of strings, i.e. sequences of symbols taken from a finite or countable set, usually referred to as an alphabet.

The general frame is the following: *we fix some constraints under which two strings are considered different and ask for the maximum number of pairwise different strings that we can have.*

Despite their purely extremal combinatorics flavor, there are numerous fields of application that stand as real motivation to the study of this type of problems: we just mention biology, cryptography, pattern recognition, information theory etc. The criteria of difference vary from one area to another but the general framework remains the same. In particular, information theory can be viewed as the study of string configurations of the above type representing asymptotic solutions to information treatment and transmission problems. One of the most important parts of information theory deals with the ways to transmit information through a noisy channel in a reliable and efficient manner. Thus the goal is to construct the maximum number of strings from an alphabet under the requirement that any two of them be "different" where what "different" means depends on the particular channel problem we are dealing with. For instance in the case of error correcting codes we require the strings to differ in many positions (i.e. to have a sufficiently large pairwise Hamming distance), while in the so called zero-error capacity problem (Shannon [57]) we require the strings to differ in special pairs of symbols depending on the channel. In the latter problem the channel is represented by a graph G on a finite set of vertices $V(G)$ and two n–length strings $x_1 \dots x_n$ and $y_1 \dots y_n$ of symbols from the set $V(G)$ are considered different according to graph G if at least one of the edges of G appears among the coordinate pairs of the strings, i.e. if there is an $i \in [n]$ such that $\{x_i, y_i\} \in E(G)$. The zero error capacity problem asks to determine the asymptotic behavior of the maximum cardinality of a set of pairwise different (or G–different) strings of length n. It is very fascinating to look at the many new and intriguing problems, applications and connections with different mathematical areas that this topic leads to. Indeed, the original problem was successfully generalized to directed graphs [47, 24], uniform hypergraphs [44], families of simple and directed graphs [11, 24], and lately it was generalized to infinite graphs as well [41] (a brief summary of the evolution of the Shannon capacity concept is given in Chapter 1 of this thesis). In the latter case the graph is defined on the whole set of natural numbers and we restrict attention to a particular type of strings of length n; permutations of the first n natural numbers. The reason for this is that if the infinite graph contains an infinite clique than for a fixed length n we can have an infinite number of pairwise G–different strings just by using the vertices of the clique, while in the remaining cases the problem itself is not really new and we will see that it exhibits the same features and difficulties as in the finite case. Thus, consequently, other requirements are needed to avoid triviality. The reason for the authors of [41] to chose permutations is that for mathematical reasons it is natural, elegant and convenient. For instance if we look at a permutation of the elements of $[n]$ as an n-length string where each of the elements of $[n]$ appears exactly once, the present concept of capacity is a natural extension of the Shannon capacity "within a given type" [12]. Furthermore, permutations have particularly interesting algebraic structures that have already been exploited when introducing so called error correcting permutation codes [9]. More generally, extending combinatorial problems from set systems to permutations, seems to be a recent trend in the literature.

Chapter 2 of this thesis deals with the study of G–different permutations for particular classes of infinite graphs [48]. Among other things we will present general bounds and show that with the exception of a few particular cases it is extremely difficult even to find an asymptotic solution to such problems. This reflects a fascinating feature of this framework: all the problems are very easy to formulate, while ironically their solutions often are very difficult. In other words, this area provides very challenging problems.

Another interesting aspect worth of study is to consider these problems together with their *forbidding* versions. For instance in the case of G-difference we require at least one of the edges of G to appear among the coordinate pairs of the two strings, whereas in the respective forbidding problem we ask for the maximum cardinality of strings such that for any two of them *none* of the edges of G should appear among the coordinate pairs. Whereas the latter problem is indeed simple, the forbidding version of the capacity problem regarding (un)directed graph families leads to many complex and interesting problems. In particular, we will focus on a special case of directed graph families. Extending this problem to the infinite case, we will introduce and study the maximum size of what we will call a *reverse free* set of permutations. Treating these problems associated with the same graph, in pairs has various advantages. On the one hand, as we will see, once we have a construction (i.e. a lower bound) for one of these problems, it is possible to infer an upper bound for the other one. It is worth mentioning that as a matter of fact these bounds are not always tight, however it might still be interesting to analyze the way those are related. On the other hand their simultaneous study sheds light on the different nature of a capacity problem and its forbidding version. First in broader terms G-difference is a local constraint in the sense that if it is satisfied by the projections of two strings into a subset of their coordinates, it is trivially satisfied by the strings themselves, while this is not true for the forbiddance relation which clearly involves some kind of global constraint. Furthermore, the latter can be considered in some sense as similarity relations, thus apart from being global they have the reflexive property, i.e. every object and itself satisfy it, while in contrast a difference relation must be irreflexive. We believe that the study of relations between such local and global constraints would help to better understand the nature of both of these problems. Chapter 3, based on the paper [22], looks at *graph difference* requirement problems from this point of view. It underscores the many close connections, parallels and also the differences between these two types of problems.

Finally, in Chapter 4 we consider some generalization of these problems where the requirements involve relations between $k > 2$ strings. It is important to note that these problems are related to the zero-error capacity problem concerning uniform hypergraphs [44]. In particular we analyze problems where we seek for the maximum number of binary strings of length n imposing requirements on fourtuples of them [40]. This problem is strongly connected with the so called problem of cancellative set families [18], which in turn is a special case of the combinatorial version of the zero error transmission problem through a binary multiplicative channel [60]. It is worth to mention that problems in extremal hypergraph theory have been studied systematically, however almost none of them has been solved even in an asymptotic sense. The cancellative problem is one of those few non–trivial problems where for an excluded configuration of size greater than two the exact exponential asymptotics is known (for another significant example see [7, 23, 35]). The fact its solution came from an information theory context [60] furnishes another good motivation to a deeper exploration of the connections between these two fields.

In this thesis we would like to emphasize the importance and challenging nature of this topic. At the same time an important goal is to point out some of the links between information theory and extremal combinatorics. These links are definitely 'two–way', and both subjects benefit. So, let's stop here and conclude borrowing a phrase of P. Cameron "... it seems that deep links in mathematics often reveal themselves in combinatorial patterns."

Chapter 1

Introduction to Graph Capacities

In this first chapter we introduce the concept of Shannon capacity and go through its various generalizations. In particular we will see how these provide a bridge between information theory and combinatorics. It is not our intention to give a complete, self contained description of every topic. Instead, the most important notions are defined and the most relevant results we will rely on, are mentioned. We refer to the bibliography for each topic, for more background information and proofs of the results. All the graphs used here are without loops and multiple edges, unless stated otherwise.

1.1 The Shannon capacity of graphs

In 1956 in his seminal paper [57] Shannon introduced zero-error problems in information theory. He considered the following scenario. Suppose we want to communicate without errors across a noisy channel by transmitting symbols from a finite input set (the alphabet). Due to the noise effect different input symbols may result in the same output at the receiving end, however for some pairs of symbols this might never occur. Shannon raised the following question:

> *"What is the maximum rate of transmission such that the receiver may recover the original message without errors? "*

Here the information rate is defined as *the maximum number of bits that can be transmitted per channel use*. We can reformulate this problem in a pure graph theory language, representing the behavior of the channel by a simple graph $G = (V, E)$ where V is the (finite) set of symbols in use and E the set of edges between **distinguishable** pairs of symbols, that is, symbols which can never be confused (i.e. result in the same output) during transmission. Now suppose for a moment we intend to communicate using single symbols. Ideally we would like to involve all the symbols of V in transmission, but since we want to communicate without errors we must restrict ourselves to use only pairwise distinguishable symbols. Thus the set of symbols we can transmit without error in a single use must induce a clique in graph G. This means that we are achieving an information rate of $\log \omega(G)$ instead of the maximum rate of $\log |V|$. However we can increase the information rate by using the channel many times, i.e. transmitting sequences of symbols in place of single symbols. To this purpose consider the case when the channel is used $n > 1$ times. The sender transmits a sequence $x_1 \dots x_n$ of input symbols and the receiver receives a sequence $y_1 \dots y_n$ of output symbols. Two sequences (strings) $\mathbf{v} = v_1 \dots v_n$ and $\mathbf{w} = w_1 \dots w_n$ from the input set V^n are **G-different** in the sense of distinguishable according to the graph G, if they are distinguishable in some coordinate; more formally if there exists a coordinate $i \in [n]$ such that $\{v_i, w_i\}$ is an edge of graph G. Again the aim is to find the maximum size of pairwise G–different strings of length n. Let $T(G, n)$ be this number. Obviously $T(G, n) = \omega(G^n)$ where G^n is the graph on vertex set V^n and edges between G–different strings. Indeed, G^n corresponds to what is usually called in literature nth *co–normal* power of G.

It is not difficult to prove that $T(G, n)$ is super–multiplicative, i.e.

$$T(G, n_1 + n_2) \geq T(G, n_1)T(G, n_2)$$

This ensures that the maximum information rate never decreases when transmitting larger sequences in an appropriate manner. The zero error capacity of a channel in the Shannon sense is defined as the maximum rate at which error–free transmission can be guaranteed. This leads to introduce the concept of **Shannon capacity** of a graph.

Definition 1.1 *Given a graph G its Shannon capacity is defined as*

$$C(G) = \lim_{n \to +\infty} \frac{1}{n} \log T(G, n)$$

The super–multiplicativity of $T(G, n)$ and Fekete's lemma (see for example [61]) ensure that the above limit actually exists.

The determination of Shannon capacity has turned out to be a very difficult problem that remains a challenge up to these days. In his paper [57] Shannon developed the first basic techniques for determining it. He observed what amounts to say, that if the chromatic number $\chi(G)$ of a graph G is equal to its clique number $\omega(G)$ then $C(G) = \log \chi(G)$ and thus for this particular class of graphs the capacity is easily determined. This inspired Berge to introduce the concept of perfect graphs. Now apart from a few cases, the Shannon capacity remains unknown even for graphs of small size. Shannon has not even settled the case of C_5, the cycle of length 5 (the smallest case of an imperfect graph), which remained open for more than 20 years until Lovász published his celebrated result [51]. However, despite all the efforts made through the years in order to determine Shannon capacity, it remains a daunting unanswered question for many graphs.

Before proceeding any further with the generalization of Shannon capacity to different kinds of graphs, let us consider a technical generalization to graphs that have a probability distribution defined on their nodes (see for example [12]). To this purpose, consider a graph $G = (V, E)$ and a probability distribution P on its vertices. For a string $\mathbf{x} = x_1 \dots x_n$ in V^n its *type* is the probability distribution $P_{\mathbf{x}}$ on the elements of V defined by

$$P_{\mathbf{x}}(a) = \frac{1}{n} \big| \{ i : x_i = a, i \in [n] \} \big|$$

for every a in V. Let $V^n(P, \varepsilon)$ denote the set of those $\boldsymbol{x} \in V^n$ whose type is approximately P, more formally for which

$$|P_{\boldsymbol{x}} - P| = \max_{a \in V} |P_{\boldsymbol{x}}(a) - P(a)| \leq \varepsilon.$$

Finally denote by $T(G, P, \varepsilon, n)$ the size of the largest subset of the $V^n(P, \varepsilon)$ whose elements are pairwise G–different. The capacity $C(G, P)$ of G within the type P is given by

$$C(G, P) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log T(G, P, \varepsilon, n).$$

This is the capacity of the channel under the restriction that the type of every element of the codebook is approximately P. We immediately recognize that the number of possible types of the strings from V^n is upper bounded by $(n+1)^{|V|}$. Hence, we obtain

$$\max_P T(G, P, \varepsilon, n) \leq T(G, n) \leq (n+1)^{|V|} \max_P T(G, P, \varepsilon, n)$$

This implies that

$$C(G) = \max_P C(G, P)$$

where the maximum is taken over all the possible distributions on V.

Thus, when dealing with Shannon capacity we can restrict ourselves to strings of the same type. We may stress here that this simple and technical generalization has been a central ingredient of many results that we will present further on.

1.2 The Sperner Capacity of directed graphs

The concept of graph capacity has been generalized from graphs to directed graphs by Körner and Simonyi in [47] and Gargano, Körner and Vaccaro in [24]. They introduced a concept of distinguishing strings according to a directed graph.

Definition 1.2 *Let $G = (V, E)$ be a loopless directed graph on a (finite) set of vertices. We say that the ordered pair of strings $(\mathbf{v}, \mathbf{w})$ is G–different if there is a coordinate $i \in [n]$, such that (v_i, w_i) is a directed edge in E. We are interested in the maximum cardinality $T(G, n)$ of a set of strings from V^n, such that any ordered pair of strings from the set is G–different.*

Note that according to the above definition, two strings $\mathbf{v}$ and $\mathbf{w}$ in V^n are G–different if there are some not necessary distinct coordinates $i \in [n]$ and $j \in [n]$, such that $(v_i, w_i) \in E$ and $(w_j, v_j) \in E$.

The **Sperner capacity** of the digraph G is defined as

$$\Sigma(G) = \lim_{n \to +\infty} \frac{1}{n} \log T(G, n)$$

Using the same argument as in the non-directed case, we can prove that $T(G, n)$ is super–multiplicative and thus the above limit actually exists.

The motivation behind the name Sperner capacity comes from the following consideration. Let G be the digraph on the set of vertices $\{0, 1\}$ and having the single directed edge $(0, 1)$. We are interested in the value of $T(G, n)$. If we look at the binary strings of length n as characteristic vectors of the subsets of a set of size n, it is immediately clear that a set of pairwise G–different strings corresponds to a Sperner family, i.e. a family no member set of which contains another. Obviously $\Sigma(G) = 1$, as by Sperner's theorem [59] we have $T(G, n) = \binom{\lfloor n/2 \rfloor}{2}$ (this statement is, however, immediate and much weaker than Sperner's result).

It is worth mentioning that Shannon capacity is indeed a special case of Sperner capacity. To show this consider a simple graph G and let G' be the associated symmetric digraph, that is the graph defined on the same set of vertices and such that between each pair of distinct vertices adjacent in G both (oppositely directed) arcs exist. A moment's thought should convince that two strings are G–different if and only if they are G'-different. Then we deduce that $C(G) = \Sigma(G')$. Furthermore, we have that the Sperner capacity of an arbitrary digraph is bounded from above by the Shannon capacity of its underlying graph. More importantly ,these two quantities are in general not equal. In other words and more precisely, Sperner capacity depends on the particular orientation of the arcs. To understand this, consider for example two different orientations of the graph K_3 on vertex set $V = \{0, 1, 2\}$: the transitively oriented triangle T_3 and the cyclically oriented one C_3.

(a) $\Sigma(T_3) = \log 3$

(b) $\Sigma(C_3) = 1$

Without much effort one can prove that $\Sigma(T_3) = \log 3$ while on the other hand it is harder to determine the Sperner capacity of the cyclic triangle. In [8] (see also [5] for a shorter proof) it is proved that we have $\Sigma(C_3) = 1$. The upper bound technique of [5] inspired by Haemers [27], was generalized by Alon [1] who proved that for any digraph D, $\Sigma(D) \leq \log \min\{\Delta^+(D)+1, \Delta^-(D)+1\}$ where $\Delta^+(D)$ (respectively $\Delta^-(D)$) is the maximum outdegree (respectively indegree) of the digraph. This upper bound was further improved by Körner, Pilotto and Simonyi in [45].

The concept of Sperner capacity turned out to be not only a fertile source of interesting new problems but also proved useful in the solution of many important open problems in extremal combinatorics as it will be seen in the next section.

1.3 The capacity of families of (un)directed graphs

So far we have seen the definition of capacity of undirected and directed graphs. In this section we show how to extend these concepts to families of undirected (directed) graphs.

1.3.1 Shannon capacity of a families of undirected graphs

Let $\mathcal{G}$ be a family of undirected graphs defined on a common finite vertex set V. Following [11] we say that two strings in V^n are $\mathcal{G}$-different if they are G–different for every G in $\mathcal{G}$. Let $T(\mathcal{G}, n)$ denote the maximum size of such a set. The following always existing limit defines **the Shannon capacity of a family of graphs**.

$$C(\mathcal{G}) = \lim_{n \to +\infty} \frac{1}{n} \log T(\mathcal{G}, n)$$

It is interesting to note that using the above definition we can now associate to a graph a new type of capacity. To this purpose consider a graph G and let $\mathcal{G}(G)$ be the family of all the single edge graphs obtained considering for each edge e of G the graph G_e with vertex set $V(G)$ and the single edge e. Finally, what we think important to mention is that this new concept of capacity differs from Shannon capacity in the following sense. In the first case the capacity involves some kind of *or*-requirements: We require that *at least one* of the edges of G occurs among their coordinate pairs, but it can be either one of the edges of G. On the other hand in the second case we deal with *and*–requirements: We require that *every* edge of G appears among their coordinate pairs.

Apart from its intrinsic interest Shanon capacity of a family of graphs has also an interpretation in information theory. It involves the so called ”compound channels”, a model which was introduced around 1960 (see for example [4, 14, 62]). Here we are concerned with guaranteeing reliable communication over any channel belonging to a given family of channels with the same finite input set. Before transmission begins, an element of this family (i.e., one of the available channels) is selected “by nature” (or our adversary, which is the same) and this choice characterizes the channel behavior for the duration of the transmission. The transmitter (but not the receiver) is assumed to be ignorant of the channel over which transmission occurs.[1] Thus, the information theory problem here consists in designing codes that will fit several channels at the same time. Now as we want to achieve zero-error communication we can represent the behavior of each channel by a graph. One can immediately recognize that in this model we are interested in strings that are pairwise G–different for each of the graphs G in the family, and thus we have to do with the capacity of a family of graphs.

However, in a purely mathematical sense, it might seem that the study of this generalization is not well justified; as it is already considerably difficult to deal with the capacity of a single graph, its extension to a family of graphs seems an impossible task. As a matter of fact the authors of [11] consider the problem from another aspect. They raise a different question: if determining the capacity of every single graph in the family is really simple, is it possible to determine without much effort the capacity of the whole family? In [26] an affirmative answer is given to this question. Roughly speaking in [26] it is proved that in order to determine the capacity of a family of graphs it is sufficient to deal with each of the graphs individually, without worrying about their interrelations. Actually, as it will be seen in what follows, they proved a stronger result concerning families of directed graphs.

[1] It is worth to note that if the receiver does not know the channel either, the situation presented is different. This was first noted by Nayak and Rose, for more details we refer to [52].

1.3.2 Sperner capacity of a family of directed graphs

The generalization of Sperner capacity to families of directed graphs follows straightforwardly. Before formally stating this definition, we want to stress that it generalizes all the definitions of graph capacities presented so far.

Definition 1.3 *Given a family $\mathcal{G}$ of directed graphs defined on a common vertex set V, we say that two strings $\mathbf{v}, \mathbf{w}$ in V^n are $\mathcal{G}$-different if they are G–different for every G in $\mathcal{G}$. More formally, for every $G \in \mathcal{G}$ there is a coordinate $i \in [n]$ such that $(v_i, w_i) \in E(G)$.*

Now in complete analogy with the non directed case let $T(\mathcal{G}, n)$ denote the maximum size of a set of pairwise $\mathcal{G}$-different strings from the set V^n. The following always existing limit defines **the Sperner capacity of the family of directed graphs** $\mathcal{G}$.

$$\Sigma(\mathcal{G}) = \lim_{n \to +\infty} \frac{1}{n} \log T(\mathcal{G}, n)$$

As we have already mentioned, the concept of Shannon capacity of a graph family is motivated as the zero-error capacity of "compound channels". Still, for Sperner capacity the original motivation comes from extremal combinatorics. In what follows we shall restrict to just a couple of illustrations to make this point (we refer to [47] for more details). However, it is worth noting that Nayak and Rose in [52], showed, that in case neither sender nor receiver know the state of the compound channel, even the latter problem can be solved in terms of Sperner capacity of families of directed graphs.

Problem 1: Let $\mathcal{G} = \{G_1, G_2\}$ where G_1 and G_2 are defined on the common vertex set $V = \{0, 1, 2\}$ and have a single edge each, precisely $E(G_1) = \{(0, 1)\}$ and $E(G_2) = \{(0, 2)\}$. We are interested in the asymptotic behavior of the maximum size of a set of $\mathcal{G}$–different strings of the same length.

Combinatorial meaning: We ask for the maximum number of pairs (A_i, B_i) of subsets of $[n]$, such that the following are verified:

1. $A_i \cap B_i = \emptyset$ for all i

2. $A_i \not\subseteq A_j \cup B_j$ for all $i \neq j$

3. $B_i \not\subseteq A_j \cup B_j$ for all $i \neq j$

The equivalence of these two problems is immediately clear if we associate to every string $\mathbf{x} \in \{0,1,2\}^n$ the sets A, B and $C = A \cup B$ as follows:

$$A = A(\mathbf{x}) = \{i : x_i = 1\}$$
$$B = B(\mathbf{x}) = \{i : x_i = 2\}$$
$$C = C(\mathbf{x}) = \{i : x_i \neq 0\}$$

Problem 2: Let $\mathcal{G}$ be the family of all single edge digraphs defined on the set of vertices $V = [k]$. We are interested in the asymptotic behavior of the maximum size of a set of k-ary $\mathcal{G}$–different strings of the same length.

Combinatorial meaning: Let X be an n-element set and let $P = (P_1, \ldots, P_k)$ and $Q = (Q_1, \ldots, Q_k)$ be two partitions of X into k disjoint classes, i.e.

$$X = \bigcup_{i=1}^{k} P_i, \quad P_i \cap P_j = \emptyset \quad \text{if } i \neq j$$

and similarly for Q. Now the two partitions P and Q are called *qualitatively* 2–*independent* if each class of one partition meets each class of the other partition, more formally if

$$P_i \cap Q_j \neq \emptyset \quad \text{for every } i \text{ and } j$$

We are interested in the largest possible size of a family of k-partitions of the n-element set X such that any two of them are qualitatively 2–independent. The problem of qualitative independence was raised in 1971 by A. Rényi [54] and had been a long standing conjecture in combinatorics (see for example [33, 37, 53]). To see how these two problems are related, first observe that we can represent a k-partition P of $X = [n]$ by its characteristic vector, i.e. a string $\mathbf{x} \in [k]^n$ such that $x_j = i$ if $j \in P_i$. It is not difficult to see that starting with a family of qualitatively 2–independent partitions and considering the characteristic vectors as suggested above, we obtain a set of pairwise $\mathcal{G}$–different strings from $[k]^n$. Furthermore the latter has the same cardinality of the original family of partitions. Being more meticulous one can note that a k-partition is represented by $k!$ different strings, but this does not make any problem because as a matter of fact two different strings that correspond to the same partition do not satisfy the criterion of $\mathcal{G}$–difference. On the other hand if we have a set of $\mathcal{G}$–different strings than we can obtain a family of partitions any two of which satisfy the condition of qualitative 2–independence except but for the case $i = j$ which may not necessary hold. To fix this apparent problem, we simply add to each string the prefix $12\dots k$. Obviously this does not affect the asymptotic behavior. Having established this we can conclude that the two problems are equivalent in the asymptotic sense.

Now as we have seen, Sperner capacity of a family is connected to many interesting problems that underscore the importance of the concept and furnish a motivation for further studies (for other examples we refer to [24, 25, 26]). The following theorem is proved in [26].

Theorem 1.4 ([26]) *Let $\mathcal{G}$ be a family of directed graphs. Then*

$$\Sigma(\mathcal{G}) = \max_{P} \min_{G \in \mathcal{G}} \Sigma(G, P)$$

where $\Sigma(G, P)$ is the Sperner capacity of the graph G within the type P.

Although the theorem does not explicitly exhibit a formula for determining the Sperner capacity of a family, its importance is reflected by the many successful applications it has found, where perhaps the most important is related to the solution of the qualitative independence problem [25]. We emphasize that it is here, in this context, that many fascinating connections between extremal combinatorics and information theory appear in many surprising ways. These connections have led to very exciting developments in both fields.

1.4 Permutation capacity of infinite graphs

The apparent success of the generalizations of Shannon capacity furnishes a good motivation for further extending this notion. In particular it would be interesting to know whether an extension to infinite graphs makes sense. To this purpose let G be an infinite graph with a countable vertex set and assume without loss of generality, $V(G)$ to be the set $\mathbb{N}$ of natural numbers. As usual, let us consider for every $n \in \mathbb{N}$ the graph G^n (some kind of power graph) whose vertex set is $\mathbb{N}^n$ (the set of all n-length strings of natural numbers) and two strings $x_1 \dots x_n$ and $y_1 \dots y_n$ from $\mathbb{N}^n$ are adjacent if there exists a coordinate $i \in [n]$ for which $\{x_i, y_i\} \in E(G)$. Now in a straightforward generalization capacity would measure the asymptotic behavior of the maximum cardinality of the clique on the nth power graph, G^n. However this in itself is not quite an interesting new problem. The point here is that if G contains an infinite clique then its capacity is trivially infinite too. Otherwise, the problem exhibits mainly the same features and difficulties as in the finite case. For example if $\omega(G) = \chi(G)$ then it is not difficult to see that the capacity is equal to $\log \chi(G)$. Now Körner and Malvenuto in [41] considered a slightly modified version of this problem. They introduced the concept of Shannon capacity of an infinite graph by restricting the attention on particular subsets of the power set $\mathbb{N}^n$: permutations of the first n natural numbers. Given an infinite graph G and a natural number n we call two permutations of the elements of $[n]$, the first n natural numbers, G–different if they map some $i \in [n]$ to adjacent vertices of G. Now for the sake of simplicity from now on we will consider permutations of $[n]$ as n-length strings that contain each element of $[n]$ exactly once. In this language, two permutations are G–different if there exists a position where the corresponding two strings contain the two endpoints of an edge of G. We denote by $T(G, n)$ the maximum cardinality of a set of pairwise G–different permutations. (Observe that if we consider the graph G^n with vertex set $\mathbb{N}^n$ and edges between G–different strings, we have that $T(G, n)$ denotes the clique number of the subgraph of G^n induced by the set of all permutations of $[n]$). The **permutation capacity** of an infinite graph G is given by the following limit

$$\rho(G) = \limsup_{n\to+\infty} \frac{1}{n} \log T(G, n)$$

As already noted, if G has an infinite clique the permutation capacity is also infinite, however the above definition gives rise to many new interesting problems that otherwise would be trivial in terms of the Shannon capacity concept. To illustrate the idea consider the following example from [41].

Problem 1*:* Consider the (semi)–infinite path L whose vertices x and y from $\mathbb{N}$ are adjacent if they are consecutive in the natural order, i.e. if $|x-y|=1$.

Now, one immediately recognizes that $\omega(L)=\chi(L)=2$ and thus the Shannon capacity $\lim_{n\to+\infty}\frac{1}{n}\log\omega(L^n)$ is equal to 1. On the other hand it is obvious that the permutation capacity is bounded from above by Shannon capacity, thus $\rho(L)\leq 1$. However, it is still an open problem to decide whether this upper bound is reached. For the best bounds on this problem see [6].

It is interesting to mention that the asymptotic growth of $T(G,n)$ can also be super–exponential. This is a trivial fact if we consider $G=K_{\mathbb{N}}$, i.e. the complete graph defined on the natural numbers. However, we will give a further less trivial example to illustrate the idea and refer to Chapter 2 for a further discussion on the asymptotic growth of $T(G,n)$.

Problem 2*:* Let G be the graph consisting on all loops on the natural numbers, more formally $V(G)=\mathbb{N}$ and $E(G)=\{\{x,x\}:x\in\mathbb{N}\}$.

Two permutations $\mathbf{x}=x_1\ldots x_n$ and $\mathbf{y}=y_1\ldots y_n$ of the set $[n]$ are G–different if and only if there exists a coordinate i for which $y_i=x_i$. The super–exponential growth derives by observing that $T(G,n)\geq(n-1)!$. This is trivially true, considering all the permutations that fix a given element. Moreover it can be easily proved that this lower bound is tight. Perhaps it might be also interesting to mention that the graph whose vertices are permutations of $[n]$ and edges between G–different permutations, is indeed the complement of the graph of permutations studied by Cameron–Ku [10] and Larose–Malvenuto [50], who showed that the above construction is the unique way to achieve the maximum clique number of $(n-1)!$.

The infinite path L belongs to the so-called *distance graphs* class. This is a class of graphs $G(D)$ that depends on a finite set $D\subseteq\mathbb{N}$ of "allowed differences" as follows: its vertices are, the natural numbers N and $\{x,y\}$ is an edge in $G(D)$ if and only if $|x-y|\in D$. When $D=\{0\}$ we have the all-loops graph described above and when $D=\{1\}$ we have the infinite path $G(D)=L$. In chapter 2 of this thesis we will further explore the permutation capacity of graphs of this type.

We should mention that all the different concepts of graph capacity presented so far (directed graph, family of directed graphs) can also be considered in the sense of permutation capacity. These generalizations provide the underpinnings from which many interesting problems arose. In this thesis we aim to present and explore some of these problems.

1.5 The Shannon capacity of uniform hypergraphs

All the generalizations of Shannon capacity we have presented up to now impose graph difference requirements on pairs of strings. Now we explore a different direction. We shall consider difference requirements over *sets* of strings. A straightford way to accomplish this is to consider hypergraphs. Indeed, Shannon capacity can be straightforwardly generalized to uniform hypergraphs [44]. To this purpose, let $H = (V, \mathcal{E})$ be a k–uniform hypergraph on m vertices, i.e. $|V| = m$ and $\mathcal{E}$ is a collection of k–element subset of V. The elements of $\mathcal{E}$ are called hyperedges.

Now in complete analogy with the undirected graph case we consider n–length strings from the set V^n and impose difference requirement on k-tuples of them. More precisely, consider k strings $\{\mathbf{v}^1, \ldots, \mathbf{v}^k\}$ from V^n, with $\mathbf{v}^l = {v^l}_1 \ldots {v^l}_n$, for every $1 \leq l \leq k$. We say that these are H*–different* if there exists a coordinate $i \in [n]$ such that $\{\mathbf{v}_i^1, \ldots, \mathbf{v}_i^k\}$ is an hyperedge of graph H. Now, a set of strings is H–different if so are every k-tuple of its strings.

Consider the hypergraph with vertex set V^n and hyperedges consisting of H–different strings. Denote this hypergraph as H^n. More formally the set of hyperedges is defined as

$$\mathcal{E}(H^n) = \big\{\{\mathbf{v}^1, \ldots, \mathbf{v}^k\} : \exists i \text{ such that } \{\mathbf{v}_i^1, \ldots, \mathbf{v}_i^k\} \in \mathcal{E}\big\}$$

Note also that in this definition the hypergraph H^n is again k–uniform, as was H. We want to find the maximum size of an H–different set of strings of length n. Let $T(H, n)$ be this number. The **Shannon capacity of a hypergraph** H is given by

$$C(H) = \limsup_{n \to +\infty} \frac{1}{n} \log T(H, n)$$

Observe that when $k = 2$ this simply reduces to the Shannon capacity of a graph, since 2–uniform hypergraphs are graphs. This alone would suffice to give an idea on the difficulty of these problems. However in hypergraph capacity problems the connections with information theory appear naturally and numerously. For instance, if H is a complete k–uniform hypergraph on m vertices, we ask for the maximum number of strings such that for every k-tuple of them, there exists a coordinate i in which all the k strings differ, i.e. assume different values. It is not difficult to see that the capacity problem is equivalent to the problem of (m, k)–perfect hashing[2] (see [19]). Indeed, it is not difficult to see that $T(H, n)$ is the maximum size of a set all k–tuples of which can be perfectly hashed by n hash functions each taking m values. The best upper bounds for the capacity of the complete k–uniform hypergraph on m vertices were obtained in [43], generalizing the approach taken in [19] and [39]. Again tools from information theory turned out to be downright useful in the study of these problems.

1.6 Structure of the thesis

The thesis consists of four chapters, and in what follows we briefly describe the contents of each of the following chapters.

Chapter 2 : The intent of this chapter is to study the largest cardinality of a set of permutations of $[n]$ any pair of which differ somewhere in a pair of adjacent vertices of a given infinite graph G. It will be seen that in some interesting special cases we will be able to completely determine this quantity. Furthermore, we compare the results in case of complementary graphs and show that permutation capacity generalizes the concept of Shannon capacity within a given type. The results of this chapter appear in

- On types of growth for graph–different permutations, J. Körner, G. Simonyi

[2] Generally speaking a perfect hash function for a set S is a hash function that maps distinct elements in S to distinct integers, with no collisions.

and B.Sinaimeri, *J. Combin. Theory Ser. A*, **116**, 713–723, 2009.

Chapter 3 : In this chapter we introduce what we will call *forbiddance problems* which arise when requiring that a pairwise difference relation between strings should never hold for any pair of strings. In particular, we will consider the forbidding version of one special case of the capacity problem for directed graph families. Extending this problem to the infinite case, we will introduce and study the maximum size of what we will call a *reverse–free* set of k-strings of distinct elements of the set $[n]$ $(k \in \mathbb{N})$. We provide rather weak bounds for the general case k, while somewhat unexpectedly we will completely solve the case $k = 3$. The results of this chapter appear in

- On sets of pairwise reverse free ordered triples, Z. Füredi, I. Kantor, A. Monti and B. Sinaimeri, *(submitted to SIAM Journal on Discrete Mathematics)*

Chapter 4 : In this chapter we consider some generalized versions of the capacity problem regarding the complete k–uniform hypergraph. In particular we focus on problems concerning the determination of the maximum cardinality of sets of binary strings of equal length, with requirements that involve triples or four–tuples of strings. In this context we introduce and study the problem of 2-cancellative set families, that we claim is a natural generalization of the cancellative concept [18]. The results of this chapter appear in

- On cancellative set families, J. Körner and B.Sinaimeri, *Combinatorics, Prob. Computing*, **16**, 767–773, 2007.

We briefly reviewed as a background to our subject, the evolution of the Shannon capacity. As we have seen its generalizations lead to many new problems and techniques, in both combinatorics and information theory. In this thesis we address some of these problems and introduce new interesting questions that we hope serve as insights in raising new research in the area.

Chapter 2

Graph–different permutations

This chapter is based on the paper [48]

In this chapter we study the largest cardinality of a set of permutations of $[n]$, any pair of which differ somewhere in a pair of adjacent vertices of a given infinite graph G. We denote this quantity by $T(G, n)$. The main goal is to initiate a systematic study of these problems for all the graphs G defined on the set of natural numbers $\mathbb{N}$. Beside this we draw attention to a particular class of infinite graphs: distance graphs. These are graphs whose vertex set is the set of natural numbers, and adjacency depends solely on the numerical difference between vertices. As it will be seen, we will give a complete solution in an interesting special case. However, despite this case, as already suggested in the previous chapter, these problems appear difficult and other cases remain unsolved. However, we consider another interesting aspect. We attempt to tease apart the different asymptotic behaviors of $T(G, n)$. We deal with various speeds of growth and try to analyze the causes that lead to a certain asymptotic behavior. In this context we are especially interested in the comparison of the results for pairs of complementary graphs, i.e. we study the relationship between the values of $T(G, n)$ and $T(\overline{G}, n)$. Finally, we find of independent interest the close relationship between our problems and the concept of Shannon capacity "within a given type".

2.1 Introduction

This chapter is devoted to the study of problems of permutation capacity. Before delving into the particular class of problems we wish to consider let us first briefly review the origins of these problems. The real inspiration for introducing permutation capacity came from the following innocent-looking mathematical puzzle of Körner and Malvenuto [41]. They call two permutations of $[n]$ *colliding* if, represented by linear orderings of $[n]$, they put two consecutive elements of $[n]$ somewhere in the same position. For the maximum cardinality $\rho(n)$ of a set of pairwise colliding permutations of $[n]$ the following conjecture was formulated.

Conjecture 2.1 ([41]) *For every $n \in \mathbb{N}$*

$$\rho(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Despite many efforts, this conjecture remains still open; for the best bounds we refer to [42] and [6].

Now, as already mentioned in the previous chapter we can formulate a similar problem in a more general context. Let G be an infinite graph whose vertex set is the set $\mathbb{N}$ of the natural numbers. We call two permutations of the elements of $[n]$, G–different if there exists a position where the corresponding two sequences contain the two endpoints of an edge of G. We denote by $T(G, n)$ the maximum cardinality of a set of pairwise G–different permutations of G. Equipped only with this definition, we are already able to pose a number of interesting and challenging problems regarding different kind of graphs, as we will see. In particular we will first restrict our attention on a special class of graphs: the so called *distance graphs*. We recall the definition of a distance graph.

Definition 2.2 *Given an arbitrary (finite or infinite) set $D \subseteq \mathbb{N}$ we define the distance graph $G = G(D)$ by setting*

$$V(G(D)) = \mathbb{N} \qquad E(G(D)) = \{\{i, i + d\} \mid i \in \mathbb{N}, d \in D\}.$$

Now, as already mentioned in Chapter 1 the problem on pairwise colliding permutations is equivalent to the G–difference problem where the graph in question is the infinite path L defined by $E(L) = \{\{i, i+1\} \mid i \in \mathbb{N}\}$. Thus, clearly, $L = G(\{1\})$. For the sake of simplicity from now on we will write

$$T(D, n) = T(G(D), n)$$

As it could be expected, the determination of $T(D, n)$ is in general a difficult task. Thus, it was somehow surprising to us to have been able to determine its exact value for every n in a non–trivial case as it will be shown in the following section. However in the other cases it still remains a difficult unsolved problem.

2.2 Superexponential growth

In the papers [41] and [42] attention was restricted to those cases where the growth of $T(G, n)$ is only exponential in n. In this section we set out to illustrate that if $D = \mathbb{N} \setminus \{1\}$ then $T(D, n)$ exhibits a super-exponential growth. What surprises most is that its determination gives a very simple formula.

Theorem 2.3

$$T(\overline{\{1\}}, n) = \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}} \quad \textit{for every} \quad n \in \mathbb{N}.$$

Proof.First we prove the upper bound.

Upper bound:

$$T(\overline{\{1\}}, n) \leq \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}.$$

To this end fix n and define $\sigma_{i,j}$ to be the permutation that exchanges the entries $i \in [n]$ and $j \in [n]$, that is, for any permutation π, $\sigma_{i,j}\pi$ differs from π only in the places where the entries i and j stand, which are exchanged. For any fixed π consider the set of permutations

$$C(\pi) := \{\sigma_{1,2}^{\varepsilon_{1,2}} \sigma_{3,4}^{\varepsilon_{3,4}} \ldots \sigma_{k,k+1}^{\varepsilon_{k,k+1}} \pi : \forall i \ \varepsilon_{i,i+1} \in \{0, 1\}\},$$

where k equals $2\lfloor n/2 \rfloor - 1$, $\sigma_{i,j}^0$ is the identity permutation, while $\sigma_{i,j}^1 := \sigma_{i,j}$. Let B be a set of permutations of $[n]$ satisfying our condition that for any pair of them there is an $i \in [n]$ they map to numbers at distance at least two and observe that the conditions imply $|C(\pi) \cap B| \leq 1$, while $C(\pi) \cap C(\pi') = \emptyset$ if $\pi, \pi' \in B, \pi \neq \pi'$. Since $|C(\pi)| = 2^{\lfloor n/2 \rfloor}$ for any π, the foregoing implies

$$|B| \leq \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}},$$

which is the claimed upper bound.

Lower bound: We will now move on to prove the lower bound. In order to accomplish this, for every n we shall explicitly construct a set of permutations satisfying the requirement. We start by the odd values of n and build our construction in a recursive manner. It will be important for the recursion that for every odd n the construction be invariant with respect to cyclic shifts. For $n = 1$ the construction consists of the identical permutation. Suppose next to have constructed

$$t_{n-2} := \frac{(n-2)!}{2^{\lfloor \frac{(n-2)}{2} \rfloor}}$$

permutations yielding a set B_{n-2} that satisfies the pairwise relation we need and has the additional property of being closed with respect to cyclic shifts. We will construct a set A_n of $\frac{n-1}{2} t_{n-2}$ permutations of $[n]$ satisfying the same pairwise condition and define B_n to be the set consisting of all the cyclically shifted versions of the elements of A_n. For an arbitrary permutation π of $[n-2]$ and $1 < j \leq n$ we define the transformations Ψ^j in the following manner. The permutation $\Psi^j \pi$ is acting on the set $[n]$,

$$\begin{aligned}
\Psi^j \pi(1) &:= n \\
\Psi^j \pi(i) &:= \pi(i-1) \quad \text{for every} \quad 1 < i < j \\
\Psi^j \pi(j) &:= n-1 \\
\Psi^j \pi(i) &:= \pi(i-2) \quad \text{for every} \quad j < i \leq n.
\end{aligned}$$

In other words, the permutation $\Psi^j \pi$ is obtained from π by prefixing n in the position preceding the first number in π and inserting $n-1$ in the j-th position of the resulting permutation. For a set A of permutations we denote by $\Psi^j(A)$ the set of the images by Ψ^j of all the permutations of $[n-2]$ belonging to A. As a last element of notation, let us denote by S^j the set of those permutations τ of $[n]$ for which $\tau^{-1}(n-2) < j$. Consider

$$A^j := \Psi^j(B_{n-2}) \cap S^j$$

and set

$$A_n := \cup_{j=2}^n A^j.$$

(One may note that $A^2 = \emptyset$ but we felt it more natural not to exclude this set from the above union.) As every permutation in A_n has n at its first position, no two of them can be cyclic shifts of each other, whence $|B_n| = n|A_n|$. Therefore in order to check that we have constructed the right number of permutations it is sufficient to verify that

$$|A_n| = \frac{n-1}{2} t_{n-2} \tag{2.1}$$

To this effect, recall that by our hypothesis the set B_{n-2} is invariant with respect to cyclic shifts. This implies that the number of those of its sequences in which a fixed element, in our case $(n-2)$, is confined to any particular subset of the coordinates is proportionate to the cardinality of the coordinate set in question, and thus

$$|A^j| = \frac{j-2}{n-2}|B_{n-2}|,$$

whence

$$|A_n| = \sum_{j=2}^{n} |A^j| = \sum_{j=2}^{n} \frac{j-2}{n-2}|B_{n-2}| = \frac{n-1}{2} t_{n-2},$$

which, substituting the value of t_{n-2}, yields

$$|A_n| = \frac{(n-1)!}{2^{\lfloor \frac{n}{2} \rfloor}}.$$

This settles our claim (2.1) and proves that B_n has the requested number of permutations.

To conclude the proof it remains to show that every pair of sequences from B_n represents a $G(\overline{\{1\}})$–different pair of permutations. We will first prove that such is the case if both sequences are from A_n. If they belong to the same A^j then this is obvious since the two permutations in such a pair must differ somewhere in those coordinates where they feature an element of B_{n-2} and thus the corresponding elements of B_{n-2}

must be different sequences. This implies, by our hypothesis, that they differ in some coordinate by strictly more than 1. If the two sequences, π and τ, do not belong to the same A^j, then we must have, say $\pi \in A^j$ and $\tau \in A^k$ with $j < k$. But then in the k-th position $\tau(k) = n-1$, while by definition, $\pi(k) < n-2$, settling this case as well.

If π and τ are two permutations that do not belong to A_n but have the value n in the same position, then they are clearly in a similar relation as their respective cyclic shifts in A_n, thus the above argument still applies.

Finally, we must prove that any two of our sequences having the symbol n in different positions also represent a $G(\overline{\{1\}})$–different pair of permutations. Now, unless the symbol n of both of the two sequences meets the symbol $(n-1)$ of the other one, we are done. Otherwise they have their respective subsequences belonging to B_{n-2} positioned in the very same coordinates and it suffices to see that these subsequences are different. For this purpose suppose that the two sequences have their symbol $(n-1)$ in the j-th and the k-th position, respectively. But then, supposing $j < k$ we can say that they must have their respective symbols $(n-2)$ in different positions since the one having its $(n-1)$ in the k-th position has its $(n-2)$ in a position belonging to the open interval (j, k) while the other one has it in the complement of the closed interval $[j, k]$ by construction. This proves our theorem for every odd n.

In order to prove our claim also for even values of n, it is enough to consider the set A_{n+1} (now $n+1$ is odd) and delete the first entry, which is $(n+1)$, from each of the permutations in this set. This way we get the right number of permutations of $[n]$ and their pairwise relations satisfy the requirement by the previous part of the proof. □

It is worth to mention that the construction presented in the proof above (in fact, a family of similar constructions that includes it) can also be described in a shorter way.

Lower bound (Second Proof). The construction is carried out by induction. Let n be odd and denote by B_n be the set of all those permutations of $[n]$ that satisfy the requirement of theorem 2.3 and furthermore in which the last three elements $n-2, n-1$ and n appear always in the same cyclic order; we can assume that it is the one illustrated in fig. 2.1. The base case $n = 3$ is trivial to establish. Now,

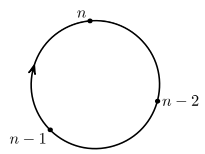

Figure 2.1: The chosen cyclic order of $n-2, n-1, n$

starting with the set B_{n-2} we will build our set of permutations B_n in the following way: For any permutation in the B_{n-2} "insert" the symbols $n-1$ and n so that in the resulting permutation (of length n) the cyclic order of $n-2, n-1$ and n is the one required. We claim that:

$$B_n = \binom{n}{2} B_{n-2}$$

Indeed, it is not difficult to convince oneself that the cardinality of the new construction of permutations of $[n]$ is as stated, since while the pair of the positions of $n-1$ and n is arbitrary (it can be chosen in $\binom{n}{2}$ ways), for any fixed pair of these positions, exactly one of their two relative orders will give the required cyclic order. This would yield the desired number of permutations claimed in the theorem. What remains to show is that the construction is correct, i.e. any two permutations in it satisfy the requirement of theorem. To this purpose consider two permutations from the construction B_n. There are three possible cases to consider:

Case I: If $n-1$ and n both have the same positions in the two permutations, then, by construction, we must have different permutations of $[n-2]$ in the other positions. Thus the requirement is satisfied by the induction hypothesis.

Case II: If $n-1$ and n are in exchanged positions in the two permutations, then, since in both cases $n-1, n$ and $n-2$ must be in the same cyclic order, we deduce that this situation can occur only if $n-2$ has a different position in these two permutations. This implies that the two permutations of $[n-2]$ they induce are different, and hence the permutations satisfy our requirement by the induction hypothesis.

Case III: If the pair of positions of $n-1$ and n is different in the two permutations then if the positions of n are different then in one of these the other permutation will have an element different from $n-1$ (otherwise the pair of positions of n and $n-1$

would coincide) and thus the requirement is satisfied. Hence, the only bad case is when both permutations put n in the same position and put $n-1$ and $n-2$ in exchanged positions. This, however, cannot happen, because we require $n-2, n-1, n$ to be in the same fixed cyclic order as in fig. 2.1. This concludes the proof. We note that the case of even n follows the same way as in the first proof. □

Before we move on, in order to simplify the forthcoming discussion let us make the following remark needed later.

Remark 1 Consider the graph whose vertex set is the set of permutations of $[n]$ and such that two permutations form an edge if and only if they satisfy the requirement we dealt with in Theorem 2.3. Denote this graph by $H_{\overline{\{1\}}}(n)$. Observe that its clique number satisfies $\omega(H_{\overline{\{1\}}}(n)) = T(\overline{\{1\}}, n)$ by definition and notice that by the proof above its chromatic number $\chi(H_{\overline{\{1\}}}(n))$ has the same value. (The sets $C(\pi)$ defined in the proof can be regarded as color classes of an optimal coloring.)

Now, with some additional argument the above theorem can be generalized and give the exact value of $T(\overline{\{q\}}, n)$ also for $q \neq 1$. We will need the following well-known lemma, the roots of which go back to Shannon [57]. We give a short proof for the sake of completeness.

Lemma 2.4 *Let $G_1, \ldots, G_k$ be graphs and let $G_1 \cdot \cdots \cdot G_k$ denote their co-normal product, i.e., the graph with vertex set $V(G_1) \times \cdots \times V(G_k)$ in which two vertices $\boldsymbol{x}, \boldsymbol{y}$ are adjacent if there is an i such that the respective i-th entries x_i, y_i of these sequences satisfy $\{x_i, y_i\} \in E(G_i)$. If $\chi(G_i) = \omega(G_i)$ holds for every i, then $\omega(G_1 \cdot \cdots \cdot G_k) = \prod_{i=1}^{k} \omega(G_i)$.*

Proof. It is easy to verify that $\omega(G_1 \cdot \cdots \cdot G_k) \geq \prod_{i=1}^{k} \omega(G_i)$ always holds. To prove the reverse inequality observe that $\chi(G_1 \cdot \cdots \cdot G_k) \leq \prod_{i=1}^{k} \chi(G_i)$. By $\omega(G_1 \cdot \cdots \cdot G_k) \leq \chi(G_1 \cdot \cdots \cdot G_k)$ the conditions $\chi(G_i) = \omega(G_i)$ imply the statement. □

The following corollary generalizes the result of theorem 2.3.

Corollary 2.5 *Let q be an arbitrary fixed natural number and let n have the form $aq+m$, where $m \in \{0,\dots,q-1\}$. Then*

$$T(\overline{\{q\}},n) = \frac{n!}{\left(2^{\lfloor \frac{a}{2} \rfloor}\right)^{q-m}\left(2^{\lfloor \frac{a+1}{2} \rfloor}\right)^{m}}.$$

Proof. Let S_n be the set of all permutations of $[n]$ represented as sequences and consider a largest possible set B_n of sequences from S_n which satisfies the requirements for $D = \overline{\{q\}}$. Let $h : \mathbb{N} \to \{0,\dots,q-1\}$ be the residue map modulo q, or, in fact, any map for which $h(k) = h(\ell)$ if and only if q divides $|k-\ell|$. For sequences $\boldsymbol{x} = x_1 \dots x_n$ extend h as $h(\boldsymbol{x}) := h(x_1)\dots h(x_n)$. Partition S_n according to the image of h, i.e., put $\boldsymbol{x}$ and $\boldsymbol{y}$ into the same partition class iff $h(\boldsymbol{x}) = h(\boldsymbol{y})$. The number of partition classes so obtained is

$$t := \frac{n!}{(a!)^{q-m}((a+1)!)^m} = \binom{n}{a,\dots,a,a+1,\dots,a+1}$$

We call the classes $W_1,\dots,W_t$. If two sequences $\boldsymbol{x},\boldsymbol{y}$ belong to different W_j's then there must be a position i for which $|x_i - y_i|$ is not divisible by q, in particular, it is not equal to q. Thus $T(\overline{\{q\}},n)$ is just the sum of the maximum possible cardinalities of sets of sequences one can find within each W_j such that each pair of these sequences satisfies the condition.

Fix any class W_j. For each $\boldsymbol{x} \in W_j$ and each position i the value $h(x_i)$ is the same by definition. Let h_i^j denote this common value. For $k \in \{0,\dots,q-1\}$ set $E_k = \{i \mid h_i^j = k\}$. Consider the subsequence of each $\boldsymbol{x} \in W_j$ given by the entries at the positions belonging to E_k. Note that the size of $|E_k|$ is either a or $a+1$. Let H_k be the following graph. Its vertex set consists of $|E_k|$-length sequences of different numbers from $[n] \cap \{\ell \mid h(\ell) = k\}$. Two such sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent in H_k iff at some coordinate i we have $|x_i - y_i| \neq q$. It is straightforward that H_k is isomorphic to the graph $H_{\overline{\{1\}}}(|E_k|)$ defined in Remark 1. Whence its clique number is $T(\overline{\{1\}},|E_k|)$, while, by Remark 1, its chromatic number has this same value. Let $\hat{H}_j$ be the graph with vertex set W_j where two vertices are adjacent if they satisfy the requirement that at some position their difference is neither 0 nor q. One easily verifies that $\hat{H}_j$ is isomorphic to the co-normal product (for the definition see Lemma 2.4) of the graphs $H_0, \ldots, H_{q-1}$, which is, by the foregoing, isomorphic to $\prod_{k=0}^{q-1} H_{\overline{\{1\}}}(|E_k|)$. We are interested in the clique number of this graph. By Lemma 2.4 and Remark 1 this value is equal to $\prod_{k=0}^{q-1} \omega(H_{\overline{\{1\}}}(|E_k|))$. Noticing that $q-m$ of the sets E_k have size a and m of them have size $a+1$, this is further equal to $\left(a!/2^{\lfloor \frac{a}{2} \rfloor}\right)^{q-m} \left((a+1)!/2^{\lfloor \frac{a+1}{2} \rfloor}\right)^{m}$ by Theorem 2.3.

The latter value is the same for all sets W_j and the number of these sets is $\binom{n}{a,\ldots,a,a+1,\ldots,a+1}$ (with a and $a+1$ appearing $q-m$ and m times, respectively). Thus we have obtained

$$T(\overline{\{q\}}, n) = \binom{n}{a, \ldots, a, a+1, \ldots, a+1} \left(\frac{a!}{2^{\lfloor \frac{a}{2} \rfloor}}\right)^{q-m} \left(\frac{(a+1)!}{2^{\lfloor \frac{a+1}{2} \rfloor}}\right)^{m}$$

$$= \frac{n!}{\left(2^{\lfloor \frac{a}{2} \rfloor}\right)^{q-m} \left(2^{\lfloor \frac{a+1}{2} \rfloor}\right)^{m}}.$$

□

2.3 Graph pairs

We find interesting to study the relationship of the values of $T(D, n)$ for pairs of disjoint sets (graphs) and their union, especially in case of pairs of complementary sets. In this section we will take a first look at these relations. To this purpose let us define

$$\phi(D, \overline{D}) := \limsup_{n \to \infty} \frac{1}{n} \log \frac{T(D, n) T(\overline{D}, n)}{n!}$$

and call it the *split strength* of the partition $\{D, \overline{D}\}$ of the natural numbers. Consider the case $D := \{1\}$. We know from [42] and [6] that that

$$2^{0.8604n} \leq T(\{1\}, n) \leq 2^n.$$

Using this in combination with Theorem 2.3 yields

Proposition 2.6

$$0.3104 < \phi(\{1\}, \overline{\{1\}}) \leq \frac{1}{2}$$

We continue with other examples. Denoting by $2\mathbb{N}$ the set of the even numbers, we would like to determine $\phi(2\mathbb{N}, \overline{2\mathbb{N}})$. To this end, notice first that

$$T(\overline{2\mathbb{N}}, n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

In fact, this easily follows, as in [41], by observing that two permutations differ in every position by an even number if and only if the even numbers occupy the same set of positions in both.

Somewhat surprisingly, $T(2\mathbb{N}, n)$ seems hard to determine and we only have some easy bounds.

Proposition 2.7

$$\frac{n!\left(\lceil \frac{n}{2} \rceil + 1\right)}{2\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq T(2\mathbb{N}, n) \leq \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}$$

Proof. The upper bound is a trivial consequence of (the upper bound part of) Theorem 1. Although the lower bound follows from the lower bound on $\kappa(K_n)$ in [42], yet for the sake of completeness we give the details without explicit reference to said paper (for more detail we refer to [42]). We consider the set $[n]$ as the disjoint union of its respective subsets of odd and even numbers. Correspondingly, we divide the coordinate set in two (with a little twist). In the first $\lceil \frac{n}{2} \rceil + 1$ coordinates we write the even permutations of the set A consisting of all the odd numbers from $[n]$ with the addition of the extra symbol $\star$. (More precisely, first we represent these $\lceil \frac{n}{2} \rceil + 1$ many symbols bijectively by the first natural numbers up to their cardinality, then extend this bijection to the permutations of both sets and consider only those permutations of the elements of A that correspond to the even permutations of the first $|A|$ natural numbers). We represent an arbitrary permutation of A in form of a sequence $\boldsymbol{x}$ and similarly let $\boldsymbol{y}$ be an arbitrary permutation of the set B of the even elements of $[n]$. We will say that $\boldsymbol{y}$ is *hooked up to* $\boldsymbol{x}$ if we replace the $\star$ in $\boldsymbol{x}$ by the first coordinate of $\boldsymbol{y}$ and concatenate the rest of $\boldsymbol{y}$ as a suffix to the resulting sequence. Let us denote by $\boldsymbol{x} \leftharpoondown \boldsymbol{y}$ the permutation of $[n]$ so obtained. Define $A \leftharpoondown B$ to be the set of all these permutations as $\boldsymbol{x}$ and $\boldsymbol{y}$ take all of their possible values. Clearly,

$$|A \leftharpoondown B| = \frac{1}{2}\left(\left\lceil \frac{n}{2} \right\rceil + 1\right)! \left\lfloor \frac{n}{2} \right\rfloor!$$

which in turn equals the claimed lower bound in the statement of the Proposition. It is very easy to see on the other hand that all the pairs of permutations from $A \leftharpoondown B$ differ by an even number in some coordinate. □

The following corollary derives immediately.

Corollary 2.8

$$0 \leq \phi(2\mathbb{N}, \overline{2\mathbb{N}}) \leq \frac{1}{2}$$

□

Next we quickly review the following immediate consequence of our hitherto results on split strength.

Proposition 2.9 *Let q be an arbitrary but fixed natural number. Then $\phi(\{q\}, \overline{\{q\}})$ is independent of the actual value of q.*

Proof. We prove more, namely that the asymptotics of $T(\{q\}, n)$ is independent of the value of q we fix and the same is true for $T(\overline{\{q\}}, n)$. For the latter it follows immediately from the formula given in Corollary 2.5.

Now we turn to $T(\{q\}, n)$. Consider the distance graph $G(\{q\})$ of the set $\{q\}$ and look at the graph it induces on $[n]$. Since the latter is isomorphic to a subgraph of P_n, the path on n vertices that the analogous distance graph $G(\{1\})$ induces on the same set, we immediately see that

$$T(\{q\}, n) \leq T(\{1\}, n). \tag{2.2}$$

In the reverse direction, we just have to observe that, for every $m \in \{0, 1, \ldots, q-1\}$, the graph $G(\{q\})$ induces an infinite path on the residue class $q\mathbb{N}+m$ of the numbers congruent to m modulo q. This implies

$$T(\{q\}, n) \geq \prod_{m=0}^{q-1} T\left(\left\lfloor \frac{n-m}{q} \right\rfloor, \{1\}\right) \tag{2.3}$$

by concatenating the respective constructions of permutations for each fixed m. Whence it is immediate that $T(\{q\}, n)$ and $T(\{1\}, n)$ have the same exponential growth rate. □

We know very little about split strength and thus there are many questions to ask. Is it always true that $\phi(D, \overline{D})$ is finite and non–negative as it seems by these examples? In order to see the greater picture, we have to look at different kind of growth rates as well.

2.4 Intermediate growth

So far we have only seen growth rates at an exponential factor away from either 1 or $n!$. We intend to show here, however, that in between growth rates are also possible. In particular, we will see that $T(D, n)$ and $T(\overline{D}, n)$ can have essentially the same growth rate, while their product is still about $n!$.

Let $ex(n)$ denote the largest exponent s for which 2^s is a divisor of n. We define

$$E := \{n \mid n \in \mathbb{N},\ ex(n) \equiv 0 (\mathrm{mod}\ 2)\}. \tag{2.4}$$

Theorem 2.10 *If n is a power of 4, then we have*

(a)

$$(\sqrt{n})!^{\sqrt{n}} \leq T(E, n) \leq \frac{n!}{(\sqrt{n})!^{\sqrt{n}}},$$

(b)

$$(\sqrt{n})!^{\sqrt{n}} \leq T(\overline{E}, n) \leq \frac{n!}{(\sqrt{n})!^{\sqrt{n}}}.$$

Proof. We prove the lower bound part of (a) first. It will be convenient to consider the elements of $[n]$ as binary sequences of length $t := \lceil \log n \rceil$, with each natural number from $[n]$ represented by its *binary expansion.* (Integer parts could be deleted by our assumption on n, moreover, we also know that t is an even number.) In fact, instead of permuting the n integers in $\{1, \ldots, n\}$, now we will permute the n numbers in $\{0, \ldots, n-1\}$. With a shift by 1, the two are obviously equivalent for our purposes. For simplicity, we will index the coordinates of the binary expansions from right to left. Hence in particular m is odd if in its binary expansion $\boldsymbol{x} = x_t x_{t-1} \ldots x_1$ the rightmost coordinate x_1 is 1 and even else. Let further $\boldsymbol{x}^{odd}$ and $\boldsymbol{x}^{even}$ denote the subsequence of the odd and the even indexed coordinates of $\boldsymbol{x}$, respectively. Finally, let $\nu(\boldsymbol{x})$ be the smallest (i. e., rightmost) index i for which $x_i = 1$. By a slight abuse of notation we will consider the various subsets of $\{0, \ldots, n-1\}$ as subsets of $\{0,1\}^t$. Quite clearly, for every $\boldsymbol{x} \in \{0,1\}^t$ we have

$$\nu(\boldsymbol{x}) = ex(\boldsymbol{x}) + 1$$

and, in particular, $\boldsymbol{x} \in E$ if and only if $\nu(\boldsymbol{x}) \equiv 1$ modulo 2. In order to prove the lower bound, let us consider the partition induced on $\{0,1\}^t$ (i.e., on $\{0, \ldots, n-1\}$) by the mapping $f : \{0,1\}^t \to \{0,1\}^{\frac{t}{2}}$ where

$$f(\boldsymbol{x}) := \boldsymbol{x}^{even} \text{ for every } \boldsymbol{x} \in \{0,1\}^t.$$

(The classes of the partition are the full inverse images corresponding to the various values of f.) It follows by construction that

$$f(\boldsymbol{x}) = f(\boldsymbol{y}) \quad \text{implies} \quad |\boldsymbol{x} - \boldsymbol{y}| \in E \tag{2.5}$$

where by the difference of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ we mean the difference in ordinary arithmetics of the natural numbers they represent. Indeed, executing the subtraction in the binary number system we are using here one sees that both $\boldsymbol{x} - \boldsymbol{y}$ and $\boldsymbol{y} - \boldsymbol{x}$ have their rightmost 1 in the position where, scanning the binary expansions of $\boldsymbol{x}$ and $\boldsymbol{y}$ from right to left, we find the first position in which they differ. Now, since $\boldsymbol{x}^{even} = \boldsymbol{y}^{even}$ by assumption, the position in question must have an odd index. In other words, $\nu(|\boldsymbol{x} - \boldsymbol{y}|) \equiv 1$ modulo 2. For every $\boldsymbol{z} \in \{0,1\}^{\frac{t}{2}}$ we denote by $S(\boldsymbol{z})$ the set of all the permutations of the elements of the full inverse image $f^{-1}(\boldsymbol{z})$ of $\boldsymbol{z}$. Thus, by our previous argument, all these permutations are pairwise $G(E)$–different. Consider the Cartesian product

$$C := \prod_{\boldsymbol{z} \in \{0,1\}^{\frac{t}{2}}} S(\boldsymbol{z}). \tag{2.6}$$

Note that the elements of C are permutations of the numbers in $\{0, \ldots, n-1\}$. The above consideration implies that C is a set of pairwise $G(E)$–different permutations. Further, observing that for every $\boldsymbol{z} \in \{0,1\}^{\frac{t}{2}}$

$$|f^{-1}(\boldsymbol{z})| = 2^{\frac{t}{2}}$$

we have

$$|C| = (2^{\frac{t}{2}})!^{2^{\frac{t}{2}}} \tag{2.7}$$

proving the lower bound in (a). (One might get a somewhat larger set by using the hookup operation instead of straightforward direct product but we do not intend to increase the complexity of the presentation for this relatively small gain here.)

Next we prove the upper bound part of (b). Notice that the set C we have constructed above has a stronger property than needed so far. In fact, in every coordinate, the absolute difference of our permutations is either 0 or else it belongs to E. This observation will be the basis for our upper bound.

Consider the auxiliary graph $H_{\overline{E}}$ the vertices of which are the permutations of $[n]$ and two are adjacent if they satisfy the requirement that at some position they have two numbers such that their difference is in $\overline{E}$. Clearly, $T(n, \overline{E}) = \omega(H_{\overline{E}})$, the clique number of this graph by definition. The above observation about C implies that its independence number $\alpha(H_{\overline{E}})$ is at least $|C|$. Note that $H_{\overline{E}}$ is vertex transitive (any permuting of the coordinates in the vertices gives an automorphism), thus by a well-known fact (cf., e.g., in [56]) its fractional chromatic number $\chi_f(H_{\overline{E}})$ is equal to the ratio of the number of vertices and the independence number. Using also that the clique number cannot acceed the fractional chromatic number (cf. [56]) we obtained

$$T(\overline{E}, n) = \omega(H_{\overline{E}}) \leq \chi_f(H_{\overline{E}}) = \frac{|V(H_{\overline{E}})|}{\alpha(H_{\overline{E}})} \leq \frac{n!}{|C|} = \frac{n!}{(2^{\frac{t}{2}})!^{2^{\frac{t}{2}}}}$$

proving the upper bound in part (b).

Exchanging the role of even and odd above we obtain the lower bound in (b) and the upper bound in (a) in a similar way. □

Theorem 2.10 shows that the investigated values of $T(E, n)$ and $T(\overline{E}, n)$ have the same growth rate at around $\sqrt{n!}$. The following statement is a straightforward generalization of the above.

Theorem 2.11 *For every rational number $\alpha \in (0, 1)$, there is a set $E_\alpha \subseteq \mathbb{N}$ such* *that for infinitely many values of n we have*

$$(n^\alpha)!^{n^{1-\alpha}} \le T(E_\alpha, n) \le \frac{n!}{(n^{1-\alpha})!^{n^\alpha}}$$

and

$$(n^{1-\alpha})!^{n^\alpha} \le T(\overline{E_\alpha}, n) \le \frac{n!}{(n^\alpha)!^{n^{1-\alpha}}}.$$

Remark 2 Notice that taking logarithm and using the estimate $\log(k!) \approx k \log k$ the above inequalities give that $\log T(E_\alpha, n)$ is about $\alpha \log(n!)$, while $\log T(\overline{E}_\alpha, n)$ is about $(1-\alpha)\log(n!)$.

Proof. Let

$$E_\alpha := \{m \mid m \in \mathbb{N},\ ex(m) \equiv 0, 1, \dots, p-1 (\mathrm{mod}\ q)\}.$$

Set $\alpha = p/q$ and suppose n is a power of 2^q. The reasoning is essentially the same as in Theorem 2.10, which is the case $\alpha = 1/2, q = 2$. Instead of $[n]$ we again permute the elements of $\{0, \dots, n-1\}$ and represent each of these numbers by their binary expansion. We collect into one group those numbers of $\{0, \dots, n-1\}$ whose binary expansion has the very same subsequence in those positions which are indexed by numbers congruent to $p+1, \dots, q$ modulo q. There are $n^{\frac{q-p}{q}} = n^{1-\alpha}$ different such groups each containing n^α different numbers. Permuting the numbers within a group we get $(n^\alpha)!$ permutations of those numbers and these are bound to differ at some position by the difference of two different numbers in the group. Such a value belongs to E_α by construction. Putting all permutations of all our groups together we obtain $(n^\alpha)!^{n^{1-\alpha}}$ different permutations altogether that not only satisfy the requirements given by the set E_α but no two of which satisfy the requirements prescribed by $\overline{E}_\alpha$. This gives the upper bound for $T(\overline{E}_\alpha, n)$ in a similar way as the upper bound on $T(\overline{E}, n)$ is proven in Theorem 2.10. The rest is also similar to what we have seen there. □

2.5 Exponential growth and Shannon capacity

In this section we return to the more familiar territory of distance graphs with finite chromatic number. The relevance of this parameter is shown by the following simple observation.

Proposition 2.12 *Let G be an infinite graph with finite chromatic number $\chi(G)$. Then*

$$T(G,n) \leq (\chi(G))^n$$

Proof. Let $c : V(G) \to [\chi(G)]$ be an optimal coloring of the vertices of G and let $c_n : V(G)^n \to [\chi(G)]^n$ be its usual extension to sequences. Notice that none of the full inverse images c_n^{-1} of the elements of $[\chi(G)]^n$ can contain two pairwise G–different permutations of $[n]$.

□

In particular, distance graphs of "rare" sets of distances have finite chromatic number. More precisely, by a result of Ruzsa, Tuza and Voigt [55], if the set $D := \{d_1, d_2, \dots d_n, \dots\}$ has the density of a geometric progression in the sense that $\liminf_{n\to\infty} \frac{d_{n+1}}{d_n} > 1$, then the distance graph $G(D)$ has finite chromatic number. Clearly, this density condition is sufficient but not necessary for the chromatic number to be finite (cf., e.g., the set of odd numbers as differences that result in a bipartite graph.)

However, for some graphs G with finite chromatic number one can give a better upper bound on $T(G,n)$. This bound is easily obtained once we realize the tight connection of our present problem with the classical concept of Shannon capacity of a graph [57]. As we have already mentioned in Chapter 1 it is easy to prove that in fact $C(G) = \max_P C(G,P)$. (for further details see [12])

In what follows we will restrict attention to graphs we call *residue graph*s. We say that an infinite graph G with vertex set $\mathbb{N}$ is a residue graph if there exists a natural number r and a finite graph $M = M(G)$ with vertex set $\{0,1,\dots,r-1\}$ such that

$$\{a,b\} \in E(G) \quad \text{if and only if} \quad \{(a)_{\text{mod } r}, (b)_{\text{mod } r}\} \in E(M)$$

Let Q be the uniform distribution on $\{0,1,\dots,r-1\}$. We have

Theorem 2.13

$$\lim_{n\to\infty} \frac{1}{n} \log T(G,n) = C(M(G),Q).$$

Proof. To prove

$$\limsup_{n\to\infty} \frac{1}{n} \log T(G,n) \geq C(M(G),Q)$$

consider, for every n those sequences $\boldsymbol{x} \in \{0,1,\ldots,r-1\}^n$ whose type Q_n satisfies

$$Q_n(a) = \frac{1}{n}|\{m \mid m \leq n, (m)_{\mathrm{mod}\, r} = a\}|$$

for every $a \in \{0,1,\ldots,r-1\}$. Let M^n be the graph whose vertices are the n–length sequences of vertices of M and whose vertices are adjacent if the corresponding sequences are M–different. For every n let C_n denote a complete subgraph of maximum cardinality M^n induces on the set of sequences of type Q_n. Notice that $\lim_{n\to\infty} \frac{1}{n} \log |C_n| = C(M(G,Q))$, because any sequence in $[V(M)]^n(Q,\varepsilon)$ can be extended to one of type $Q_{n'}$, where $n \leq n' \leq n + 2|V(M)|\varepsilon n$, i.e., by adding at most $2|V(M)|\varepsilon n$ new coordinates. This shows $|C_{n'}| \geq \omega(G, Q_n, \varepsilon, n)$ and since $|V(M)|$ is constant and ε can be chosen arbitrarily small, it proves that $\frac{1}{n}\log|C_n|$ tends to $C(M(G,Q))$ as n goes to infinity. Now for any fixed n and to any sequence $\boldsymbol{x} \in V(C_n)$ we associate a permutation of $[n]$ by replacing the occurrences of a in the sequence by the different numbers congruent to a modulo r, in a strictly increasing order. The result is a set of permutations which is G-different and has cardinality $|C_n|$. With the observation above this proves the desired inequality.

For the reverse inequality let n be a multiple of r and consider any construction achieving $T(G,n)$, i.e., a set of permutations of the elements of $[n]$ that are pairwise G-different, while the cardinality of the set is $T(G,n)$. Substitute the occurrence of the number i in each of these permutations by the unique $j \in \{0,\ldots,r-1\}$ which is congruent to it modulo r. Doing this for all $i \in [n]$ we get $T(G,n)$ different sequences of vertices of M each having type Q that actually form a clique in M^n. □

The following corollary is immediate.

Corollary 2.14

$$\lim_{n\to\infty} \frac{1}{n} \log T(G,n) \leq C(M(G)).$$

□

Let us consider the following

Example Let G have vertex set $\mathbb{N}$ and set

$$\{a, b\} \in E(G) \quad \text{if} \quad |a - b| \equiv 1 \text{ or } 4 \pmod 5.$$

As an easy consequence of Lovász' celebrated formula [51] for the Shannon capacity of the pentagon graph we obtain, using the last theorem (and also that the Shannon capacity of C_5 is obtained by sequences the type of which is the uniform distribution), that

$$\lim_{n\to\infty} \frac{1}{n} \log T(G, n) = \frac{1}{2} \log 5.$$

It is an easy observation that for any graph M and any rational probability distribution P on its vertex set one can construct (by simply substituting each vertex by an independent set of appropriate size) a graph $\hat{M}$ for which $C(M, P) = C(\hat{M}, Q)$, where Q is again the uniform distribution. It is then easy to construct an infinite graph G, which is a residue graph with respect to $\hat{M}$ and thus the asymptotics of $T(G, n)$ is in an analogous relationship with the capacity $C(M, P)$ as the one expressed in Theorem 2.13. Taking into account the remark that $C(M)$ can be expressed as the maximum of the values $C(M, P)$ over P, we can conclude that the class of problems asking for the asymptotics of $T(G, n)$ for various infinite graphs G contains the Shannon capacity problem of all such graphs for which Shannon capacity is attained as the capacity within a type for some rational distribution.

Chapter 3

Forbiddance problems

This chapter is based on the paper [22]

In this chapter we will see that, for a fixed graph (or family of graphs), a new type of problem arises when the pairwise relation between strings required to hold for the capacity problem is now forbidden to hold for any pair of strings in a possibly large subset of all the strings. In particular, in the capacity problem for a graph family, this approach leads to a new and challenging collection of problems. Following [38] we will explain the exact asymptotic solution regarding the special case of a family of single edge graphs defined by the complete graph on three vertices. Further we will consider the forbidding version of one particular case of the capacity problem for directed graph families. Extending this problem to the infinite case, we will introduce and study the maximum size of what we will call a *reverse–free* set of permutations. However, this problem seems difficult and even the asymptotic behavior remains unknown. Hence, we will focus on *partial permutations* of $[n]$, i.e. for some integer k we consider k-strings of distinct elements of the set $[n]$ (in short, k-strings). We provide rather weak bounds for the general case k, while somewhat unexpectedly we will completely solve the case $k = 3$.

3.1 Introduction to the forbiddance concept

Through changing, adding to, or removing one or more characteristics of the initial instance, every problem becomes a possible seed for new problems. Most new problems are inspired either by older ones, as described above, or by contexts that direct the problem poser in a particular direction. Faced with an old question, we can try numerous changes that would lead to new investigations. This is what we will do in this chapter. We will tackle a new class of problems originated in a question posed by A. Monti [22] and further considered in a more general and unifying context by Körner [38]. These problems arise when forbidding that the difference relation in one of the previous capacity problems be ever satisfied. Roughly speaking, the general framework for all the problems we have presented was the following: we require that for any pair of strings there is a coordinate which satisfies a given condition. Here the idea is to consider the following modified version: we require that for any pair of strings the condition *never* be satisfied by any of the coordinate pairs. We will better illustrate the idea by an example. To this purpose consider as a starting point the Shannon capacity problem.

Shannon capacity problem: Given a graph G we are looking for the maximum number of n–length strings of symbols in $V(G)$ such that for any two strings $\mathbf{x}$ and $\mathbf{y}$ in the set there *exist* a coordinate $i \in [n]$ for which $\{x_i, y_i\} \in E(G)$.
Now negating the difference requirement we obtain the so called *forbidding version* of this problem:

Forbidding version: Given a graph G we are looking for the maximum number of n–length strings of symbols in $V(G)$ such that for any $\mathbf{x}$ and $\mathbf{y}$ in the set for **none** of the coordinates $i \in [n]$ we have that $\{x_i, y_i\}$ is an edge of G.

Let $N(G, n)$ denote the largest cardinality of a set of strings from $V(G)^n$ that satisfies the requirement of the latter problem. The **forbiddance** of a graph is defined as follows:

$$F(G) = \limsup_{n\to+\infty} \frac{1}{n} \log N(G, n)$$

It is not difficult to see that the new problem is indeed really simple. To this purpose consider a set $\mathcal{C} \subseteq V(G)^n$ such that for any two strings from the set, none of the edges of G appears among their coordinate pairs. Now, consider for a fixed coordinate i, the set S_i whose elements are the projections of the strings of $\mathcal{C}$ onto i. Formally,

$$S_i = \{x_i : \mathbf{x} \in \mathcal{C}, \mathbf{x} = x_1 \ldots x_n\}$$

It is immediately clear that there are no edges between any two symbols of S_i, otherwise the respective strings could not satisfy the required condition. Thus, S_i forms an independent set in G and hence $|S_i| \leq \alpha(G)$. A moment of thought should convince that $N(G,n) = |\mathcal{C}| = \left[\alpha(G)\right]^n$. Thus, in this case the graph forbiddance problem is indeed trivial. However it suffices to give an idea of the type of the problems we are going to explore.

Before we move on, it is useful to observe the following. First, as we have seen for a graph G, we have defined two concepts: its capacity and its forbiddance. We show now that in both of these cases we look for the maximum clique in some kind of power graph of G. To this purpose we will restate the definitions of two types of graph operations that are also well known in literature (see for example [29])

Definition 3.1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. Their*

- ***conormal product*** *(disjunctive product, OR product) $G_1 \cdot G_2$ is defined as the graph with vertex set $V = V_1 \times V_2$ and there is an edge between two distinct vertices (v_1, v_2) and (w_1, w_2) if $\{v_i, w_i\}$ is an edge in E_i for at least one i in $\{1, 2\}$.*

- ***normal product*** *(strong product, AND product) $G_1 \times G_2$ is defined as the graph with vertex set $V = V_1 \times V_2$ and there is an edge between two distinct vertices (v_1, v_2) and (w_1, w_2) if for every i in $\{1, 2\}$ we have either $v_i = w_i$ or $\{v_i, w_i\}$ is an edge in E_i.*

Now from the definition it is clear that the Shannon capacity of a graph G is concerned with finding the maximum clique in the conormal powers of G. Furthermore, we can trivially argue that in case of the forbiddance of a graph G we are concerned with finding the maximum clique in the normal powers of the complementary graph $\overline{G}$ of G.

A further interesting aspect of the problems of capacity and forbbidance of a graph is their completely different nature. It is clear that in Shannon capacity problems the requirements are local, i.e. if a condition is satisfied by the projections of two strings onto a subset of their coordinates, it is trivially satisfied by the strings themselves, while this is clearly false for forbiddance which involves a global constraint.

As we have seen, the forbidding version of the capacity of a graph G leads to a trivial problem, however the situation is completely different if we consider forbiddance and capacity of a family of undirected graphs.

3.2 The forbbidance of undirected graph families

In this section we consider the forbidding version of the capacity problem for undirected graph families. In particular, we focus on the special case of a family of single edge graphs defined by a given graph G.

The capacity problem of a single edge graph family: Given a graph G we look for the maximum number of n–length strings of symbols in $V(G)$ such that for any two strings $\mathbf{x}$ and $\mathbf{y}$ in the set every edge of G appears among the coordinate pairs. More formally, for any edge $e \in E(G)$ there *exists* a coordinate $i \in [n]$ for which $\{x_i, y_i\} = e$.

Now the forbidding version of the problem is the following:

Forbidding version: Given a graph G we look for the maximum number of n–length strings of symbols in $V(G)$ such that for any $\mathbf{x}$ and $\mathbf{y}$ in the set **not all** the edges of G appear among their coordinate pairs.

The following defines the forbiddance of the single edge family $\mathcal{G}(G)$ of a graph G

$$F(\mathcal{G}(G)) = \limsup_{n \to +\infty} \frac{1}{n} \log N(\mathcal{G}, n)$$

Determining the $F(\mathcal{G}(G))$ is absolutely non trivial. To illustrate this, we will consider the case when $G = K_3$.

3.2.1 The forbiddance of $\mathcal{G}(K_3)$

In this section, following [38], we will exactly determine the forbiddance of $\mathcal{G}(K_3)$.

Theorem 3.2 ([38])

$$F(\mathcal{G}(K_3)) = 1$$

Proof. Let $\mathcal{G} = \mathcal{G}(K_3)$ be the family of the single edge graphs of the complete graph K_3, defined on the set of vertices $V = \{0, 1, 2\}$. Observe that in this forbiddance problem we are interested in the maximum cardinality of a set of ternary strings such that for any two strings in the set not all of the three edges $\{\{0, 1\}, \{0, 2\}, \{1, 2\}\}$ appear among their coordinate pairs. We begin by proving the lower bound.

Lower Bound: The lower bound follows trivially by observing that the set $\mathcal{C} = \{0, 1\}^n$ of all binary strings of length n satisfies the requirement of this forbiddance problem. Hence, $N(\mathcal{G}, n) \geq 2^n$ which yields the desired lower bound.

Upper Bound: Let $n \in \mathbb{N}$ and let $\mathcal{C}$ be a set of strings in $\{0, 1, 2\}^n$ that satisfies the forbiddance requirements. Furthermore, let $\mathcal{C}$ attain the maximum cardinality $N(\mathcal{G}, n)$. We partition $\mathcal{C}$ in classes of strings of the same type. To this purpose we will first recall some definitions already presented in Chapter 1. Recall that the type of a string $\mathbf{x} = x_1 \dots x_n$ in V^n is the probability distribution $P_{\mathbf{x}}$ on the elements of V defined by

$$P_{\mathbf{x}}(a) = \frac{1}{n}\big|\{i : x_i = a, i \in [n]\}\big|$$

for every a in V. Now given a distribution P over the set of vertices V, denote by V_P^n the set of the strings $\mathbf{x} \in V^n$ whose type is P, i.e.

$$V_P^n = \{\mathbf{x} \in V^n : P_{\mathbf{x}} = P\}$$

Note that not every probability distribution on V can be realized by the type of a string of length n. We are interested on those distributions P for which $V_P^n \neq \emptyset$. Observing that the number of possible distributions is upper bounded by $(n+1)^{|V|-1}$, it is clear that

$$\begin{aligned}|\mathcal{C}| &\leq \sum_{P: V_P^n \neq \emptyset} |\mathcal{C} \cap V_P^n| \\ &\leq (n+1)^{|V|-1} \max_{P: V_P^n \neq \emptyset} |\mathcal{C} \cap V_P^n| \end{aligned} \tag{3.1}$$

At this point, fix a probability distribution P for which the set V_P^n is not empty. Let $N(\mathcal{G}, P, n)$ denote the size of the largest subset of the V_P^n that satisfies the requirements of the forbiddance problem regarding the family $\mathcal{G}(G)$. Obviously $|\mathcal{C} \cap V_P^n| \leq N(\mathcal{G}, P, n)$. Next we will provide an upper bound for $N(\mathcal{G}, P, n)$. To this purpose, consider the graph $D_P(n)$ whose vertex set is the set V_P^n of the n–length strings of type P, and two strings are adjacent if and only if they satisfy the forbiddance requirement of our problem. Thus $|V(D_P(n))| = |V_P^n|$ and furthermore it is easy to see that the graph $D_P(n)$ is vertex transitive[1]. Hence by a double counting argument we have that

$$\alpha\big(D_P(n)\big)\omega\big(D_P(n)\big) \leq |V(D_P(n))| \tag{3.2}$$

At this point observe that by definition

$$N(\mathcal{G}, P, n) = \omega\big(D_P(n)\big) \tag{3.3}$$

It remains to determine $\alpha\big(D_P(n)\big)$. To this purpose, consider the problem of determining the capacity of the family $\mathcal{G}$. Observe that two strings are not adjacent in $D_P(n)$ if and only if those are $\mathcal{G}$–different. Thus, it is clear that

$$\alpha(D_P(n)) = T(\mathcal{G}, P, n). \tag{3.4}$$

where $T(\mathcal{G}, P, n)$ is the maximum the size of the largest subset of V_P^n, whose elements are pairwise $\mathcal{G}(G)$–different. Combining (3.2)–(3.4) we obtain

$$N(\mathcal{G}, P, n) \leq \frac{|V_P^n|}{T(\mathcal{G}, P, n)} \tag{3.5}$$

[1]Recall that a graph is vertex–transitive if every vertex can be mapped to any other vertex by some automorphism of the graph.

It is not difficult to prove the following bound (see Lemma 2.3 on page 30 in [13]).

$$|V_P^n| \leq 2^{nH(P)} \tag{3.6}$$

Now considering the inequalities in (3.1), (3.5), (3.6) we have that

$$\begin{aligned} F(\mathcal{G}(K_3)) &= \limsup_{n\to\infty} \frac{1}{n} \log N(\mathcal{G}, n) \\ &= \limsup_{n\to+\infty} \frac{1}{n} \log \left[(n+1)^{|V|-1} \max_{P:V_P^n \neq \emptyset} N(\mathcal{G}, P, n) \right] \\ &\leq \limsup_{n\to+\infty} \frac{1}{n} \log \left[(n+1)^{|V|-1} \max_{P:V_P^n \neq \emptyset} \frac{2^{nH(P)}}{T(\mathcal{G}, P, n)} \right] \end{aligned}$$

Now, we will use the following result which is proved in [26].

Lemma 3.3 ([26])

$$C(\mathcal{G}(G), P) = \min_{\{a,b\}\in E(G)} \left\{ \big[P(a) + P(b)\big] h\Big(\frac{P(a)}{P(a) + P(b)}\Big) \right\}$$

Hence, using the the above lemma we derive that

$$F(\mathcal{G}(K_3)) \leq \max_P \left[H(P) - \min_{\{a,b\}\in E(G)} \big[P(a) + P(b)\big] h\left(\frac{P(a)}{P(a) + P(b)} \right) \right]$$

Now, by the chain rule we have that

$$H(P) - \big[P(a) + P(b)\big] h\left(\frac{P(a)}{P(a) + P(b)} \right) = h\big(P(c)\big)$$

where $\{a, b, c\} = \{0, 1, 2\}$. Thus,

$$\begin{aligned} F(\mathcal{G}(K_3)) &\leq \max_{P:V_P^n \neq \emptyset} \left\{ h\big(P(0)\big), h\big(P(1)\big), h\big(P(2)\big) \right\} \\ &\leq 1 \end{aligned}$$

where the last inequality derives from the fact that $h(x) \leq 1$ for all $x \in [0, 1]$ (the binary entropy funtion is given by $h(x) = -x \log x - (1-x) \log{(1 - x)}$). This concludes the proof. □

3.3 Forbiddance problems : Other cases

It is possible to define a forbidding version for all the capacity problems presented so far. In particular, forbiddance problems can be formulated in a straightforward way for families of directed finite graphs. Again we will focus on a specific case. To this purpose consider a finite undirected graph G and let $\vec{\mathcal{G}}(G)$ be the family of all directed graphs obtained by considering all possible orientations of G. The following is the capacity problem of directed graph families in this case.

The Sperner capacity problem of the family $\vec{\mathcal{G}}(G)$: Given a graph G we look for the maximum number of n–length strings of symbols in $V(G)$ such that for any two strings $\mathbf{x}$ and $\mathbf{y}$ and any orientation of the edges of G there exist two different coordinates i and j such that (x_i, y_i) and (y_j, x_j) are both arcs of the graph.

As always we will say that the strings $\mathbf{x}$ and $\mathbf{y}$ are $\vec{\mathcal{G}}(G)$–different. Now, observe that this problem is equivalent with requiring that for any two strings $\mathbf{x}$ and $\mathbf{y}$ there exist two different coordinates i and j such that (x_i, y_i) and (y_j, x_j) are arcs of the graph and furthermore $(x_i, y_i) = (y_j, x_j)$. In other words, we require the same arc to appear in both directions among the coordinate pairs of the two strings, considered in some arbitrary but fixed order (see fig.3.1).

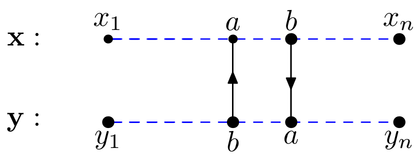

Figure 3.1: The edge $\{a, b\}$ appears in both directions.

In order to see that the two formulations are indeed equivalent, suppose on the contrary that there are two $\vec{\mathcal{G}}(G)$–different strings $\mathbf{x}$ and $\mathbf{y}$ which do not satisfy the above property. Consider the list of the ordered coordinate pairs $(x_1, y_1), \ldots, (x_n, y_n)$. Now, by assumption there are no two ordered pairs among them that represent the same edge in different directions. This means that the ordered pairs (x_i, y_i) whose elements are the endpoints of an edge of G form a subset of the set of arcs for a member $G' \in \vec{\mathcal{G}}(G)$. Hence, it is clear that $\mathbf{x}$ and $\mathbf{y}$ cannot be $\vec{\mathcal{G}}(G)$–different which is a contradiction an thus our claim is true.

Given a graph G, in the forbidding version of the above problem, we have to require that for any $\mathbf{x}$ and $\mathbf{y}$ in the set no edge of G appears in both directions among the ordered coordinate pairs (x_i, y_i). In this chapter we will focus on a slightly modified version of this problem, as we extend it to the case of an infinite graph G. In this case, as we have already done, we deal with strings that are permutations of the first n natural numbers. In particular we will focus on the case of the complete infinite graph, i.e. $G = K_{\mathbb{N}}$. Let us formulate the problem we are interested in:

Forbiddance problem: We ask for the maximum number of permutations of $[n]$, such that for any two of them π, ρ there are no two coordinates $i, j \in [k]$ for which $(\pi_i, \pi_j) = (\rho_j, \rho_i)$.

We call two such permutations *reverse–free*. This problem seems to be difficult and even the asymptotic behavior remains unknown. Indeed, we do not even know if the asymptotic growth is super–exponential or exponential. Thus, we introduce a further variation: we consider *partial permutations* of $[n]$, i.e. for some integer $k \leq n$ we consider ordered k-strings of distinct elements of the set $[n]$. For the sake of simplicity in this chapter we will refer to them as k–strings. In the next section we will focus on this case. We will provide bounds for the general case k. Somewhat unexpectedly we will be able to completely solve the case $k = 3$. The results of the rest of this chapter appear in [22].

3.4 Reverse–free k–strings

In this section we introduce and study pairwise reverse–free sets of permutations. We will present some general bounds on the maximum cardinality of these sets.

Definition 3.4 *Let k and n be natural numbers and $k \leq n$. Two k–strings $\mathbf{x} = x_1, \ldots, x_k$ and $\mathbf{y} = y_1, \ldots, y_k$ of distinct elements of the set $[n]$ are called reverse-free if there are no two coordinates $i, j \in [k]$ such that $(x_i, x_j) = (y_j, y_i)$.*

In other words two k–strings are reverse–free if they don't have the same couple of elements in the same couple of coordinates in reversed order. Furthermore a set $\mathcal{C}$ of k–strings is called *pairwise reverse–free* if so are any two of its k–strings. Let $M(n,k)$ be the maximum cardinality of a set of pairwise reverse–free k–strings from $[n]$. It seems very difficult to determine the exact value of $M(n,k)$, thus we concentrate in estimating its asymptotic behavior as $n \to +\infty$ for k fixed. In particular we are interested in the asymptotic density:

$$f(k) = \lim_{n\to+\infty} \frac{M(n,k)}{k!\binom{n}{k}} \tag{3.7}$$

First note that the limit in (3.7) actually exists since the ratio $M(n,k)/\big(k!\binom{n}{k}\big)$ is monotonically non-increasing in n. Indeed, let $n_2 > n_1$ and let M be a pairwise reverse–free set of k–strings from $[n_2]$ attaining the maximum cardinality $M(n_2,k)$. Now we 'double count' the pairs $(A,\mathbf{x})$ where A is a n_1-subset of $[n_2]$ and $\mathbf{x}$ is a k-tuple of elements of A that belongs to $\mathcal{C}$.

- Let as fix $A \subseteq \binom{[n_2]}{n_1}$. There are at most $M(n_1,k)$ strings $\mathbf{x} \in \mathcal{C}$ whose elements are in A.

- On the other hand any $\mathbf{x} \in \mathcal{C}$ determines k elements of A and thus the number of the possible choice of A are $\binom{n_2-k}{n_1-k}$.

Thus we have that:

$$\binom{n_2-k}{n_1-k} M(n_2,k) \leq \binom{n_2}{n_1} M(n_1,k)$$

Using the identity

$$\binom{n_2-k}{n_1-k}\binom{n_2}{k} = \binom{n_2}{n_1}\binom{n_1}{k}$$

the proof follows trivially.

3.4.1 Bounds on $M(n,k)$

Here we will prove the following bounds on $M(n,k)$.

Theorem 3.5

$$\binom{n}{k} \leq M(n,k) \leq \frac{k!\binom{n}{k}}{2^{\lfloor \frac{k}{2} \rfloor}}$$

Proof.

Lower Bound: Observe that all the increasing sequences of k elements from $[n]$ form a set of pairwise reverse–free k–strings. This is indeed the case as if there were two k–strings $\mathbf{x}, \mathbf{y}$ such that for two coordinates $i, j \in [k]$ we have $x_i = y_j$ and $x_j = y_i$ this would imply that the elements x_i and x_j appear in the k–strings in two different orders. Obviously one of the orders must be necessary the decreasing but this cannot be as the k–strings are increasing. Thus, we have

$$\binom{n}{k} \leq M(n,k) \tag{3.8}$$

Upper Bound: Let $\mathcal{C}$ be a set of pairwise reverse–free k–strings from the set $[n]$, such that $|\mathcal{C}| = M(n,k)$. Let $\sigma = \left\{(2i-1, 2i) : 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor\right\}$. In other words, σ is a set of $\left\lfloor \frac{k}{2} \right\rfloor$ disjoint couples of consecutive coordinates. Denote by σ_i the couple $(2i-1, 2i)$.
Let $\mathbf{x} = x_1 \ldots x_k$ be a k-tuple from $[n]$. Given $\sigma' \subset \sigma$ we obtain a new k-tuple $\mathbf{x}'$ starting from $\mathbf{x}$ and exchanging the elements at positions $2i-1$ and $2i$ for each $\sigma_i \in \sigma'$ (see figure 3.2).

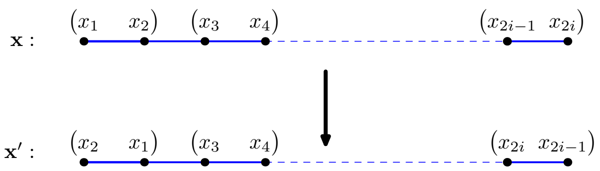

Figure 3.2: Let $\sigma = \{\sigma_1, \sigma_{2i-1}\}$, the k–string $\mathbf{x}'$ is obtained by $\mathbf{x}$ as explained.

Let $\mathcal{C}_{\mathbf{x}}$ be the set of k–strings obtained from $\mathbf{x}$ according to any subset of σ. Obviously $\mathcal{C}_{\mathbf{x}}$ is of cardinality $2^{\lfloor \frac{k}{2} \rfloor}$ and furthermore at most one of the k–strings of

$\mathcal{C}_{\mathbf{x}}$ belongs to $\mathcal{C}$ (as the couples σ_i are disjoint there are no two k–strings in $\mathcal{C}_{\mathbf{x}}$ that satisfy the reverse–free property). Also it is not difficult to see that for any $\mathbf{x} \neq \mathbf{y}$ in $\mathcal{C}$ the sets $\mathcal{C}_{\mathbf{x}}$ and $\mathcal{C}_{\mathbf{y}}$ are disjoint. Hence,

$$M(n,k) \leq \frac{k!\binom{n}{k}}{2^{\lfloor \frac{k}{2} \rfloor}} \tag{3.9}$$

This concludes the proof. □

We want to remark that from the inequalities (3.8) and (3.9) we obtain, for any fixed k,

$$\frac{1}{k!} \leq f(k) \leq \frac{1}{2^{\lfloor \frac{k}{2} \rfloor}} \tag{3.10}$$

Clearly if $k = 2$ the bounds are tight and we have $f(2) = 1/2$. Nevertheless when $k > 2$ it seems complicated to establish the exact value of $f(k)$. In the rest of this chapter we will be mainly concerned with the case of pairwise reverse–free sets of 3–strings. We will establish the exact value for $f(3)$. Furthermore, exploiting this case, we will also be able to improve the general upper bound for $M(n,k)$.

3.5 Reverse–free 3–strings

In this section we study the problem of the maximum set of pairwise 3–strings. In order to avoid unnecessarily complex notation, as in this section we will always deal with 3–strings we will refer to them simply as strings. The following theorem determines the exact result for $M(n,3)$.

Theorem 3.6

$$\frac{5}{24}n^3 - \frac{1}{2}n^2 - O(n \log n) \leq M(n,3) \leq \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n$$

where equality holds when n is a power of 3.

In the next two subsections we will prove the above theorem.

3.5.1 An iterated construction of reverse–free 3–strings

Theorem 3.7 *For any $n \in \mathbb{N}$, we have*

$$M(n,3) \geq \frac{5}{24}n^3 - \frac{1}{2}n^2 - O(n \log n). \tag{3.11}$$

If n is a power of 3, we have

$$M(n,3) \geq \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n. \tag{3.12}$$

***Proof*.** We build our construction in a recursive manner. For $n = 3$ it is easy to see that two strings on three elements are reverse–free if and only if one is a cyclic shift of the other. Thus, we have $M(3,3) = 3$ and optimal constructions are $\{123, 231, 312\}$ and $\{213, 132, 321\}$. Now consider $n > 3$ and partition $[n]$ into three disjoint sets A, B and C of cardinalities a, b and c respectively. We classify a string by determining in which of these three sets each one of its elements belongs. To this end we introduce the concept of *form* of a string.

Definition 3.8 *Given a string $\mathbf{x} = x_1x_2x_3$, we say that it has the form $X_1X_2X_3$ where $X_i \in \{A, B, C\}$ for $1 \leq i \leq 3$, if $x_1 \in X_1$, $x_2 \in X_2$ and $x_3 \in X_3$.*

Consider now the following list L of forms:

$$L = \begin{cases} AAA \\ BBB \\ CCC \\ AAB \\ AAC \\ ABB \\ CBB \\ CAC \\ CBC \\ ABC \\ CAB \end{cases}$$

Note that the strings above do not have a pair of distinct elements in reversed order. In fact one can consider L as a maximal set of pairwise reverse–free strings on three elements if we allow repetition of symbols and tolerate pointwise coincidences in two coordinates. All the strings of our construction will have a form from list L. Consider the sets $\mathcal{C}_X$, for every $X \in \{A, B, C\}$, of reverse–free strings of distinct elements from the set X attaining maximum cardinality and let $\mathcal{C}'$ be the union of these three sets. Hence, $\mathcal{C}'$ is a set of reverse–free strings of the form AAA, BBB and CCC of maximum size. Next we add to the set $\mathcal{C}'$ strings of other forms from the list L conserving the pairwise reverse–free property of the set. To this purpose define a set $\mathcal{C}''$ in the following way:

The string $x_1x_2x_3$ of the form $X_1X_2X_3$ belongs to $\mathcal{C}''$ iff the following two conditions hold:

(a) $X_1X_2X_3$ is in $L \setminus \{AAA, BBB, CCC\}$

(b) if there exist two coordinates i, j with $i < j$ such that $X_i = X_j$ then one of the following cases must occur

- $(b.1)$ there is a string $y_1y_2y_3 \in \mathcal{C}'$ such that $(y_i, y_j) = (x_i, x_j)$
- $(b.2)$ $x_i < x_j$ and there is no string $y_1y_2y_3 \in \mathcal{C}'$ such that $(y_i, y_j) = (x_j, x_i)$

Now set $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$. We claim that $\mathcal{C}$ is a set of pairwise reverse–free strings of $[n]$. Indeed, suppose to the contrary that there exist $\mathbf{x} = x_1x_2x_3$ and $\mathbf{y} = y_1y_2y_3$ in $\mathcal{C}$, of the form $X_1X_2X_3$ and $Y_1Y_2Y_3$ respectively, such that there exist two coordinates $i < j$ for which $(y_i, y_j) = (x_j, x_i)$. We have to consider two cases :

- $X_i \neq X_j$: This means that $(X_i, X_j) = (Y_j, Y_i)$ implying that $\mathbf{x}$ and $\mathbf{y}$ cannot both be of a form in L as no two of these latter forms will reverse a pair involving distinct elements. Thus we have a contradiction with the fact that by construction all the strings in $\mathcal{C}$ are of a form in L.

- $X_i = X_j$: First note that none of the strings $\mathbf{x}$ and $\mathbf{y}$ can be in $\mathcal{C}'$ as it is a pairwise reverse–free set and moreover condition (b) assures that every string of $\mathcal{C}''$ is reverse–free with any string of $\mathcal{C}'$. It remains to consider the case when both $\mathbf{x}$ and $\mathbf{y}$ belong to $\mathcal{C}''$. In this case it is easy to check that condition $(b.2)$ cannot hold for both strings. Let $\mathbf{x}$ be the string for which condition $(b.1)$ holds. This implies that $\mathbf{y}$ must have a reversed pair with a string in $\mathcal{C}'$. Hence again we arrive at a contradiction.

It remains to compute the cardinality of $\mathcal{C}$. First note that clearly

$$|\mathcal{C}| = |\mathcal{C}'| + |\mathcal{C}''| \tag{3.13}$$

and recalling that the cardinalities of the sets A, B and C are respectively a, b and c we have

$$|\mathcal{C}'| = M(a,3) + M(b,3) + M(c,3) \tag{3.14}$$

In order to estimate $|\mathcal{C}''|$ note that the number of strings produced by a form $X_1X_2X_3$ with all X_i's different is exactly abc. Now consider a form AAB. Observe that by condition (b) for every pair $\{x, y\}$ of elements from A and for every $z \in B$, one of the two strings xyz and yxz belongs to $\mathcal{C}'$. Thus, we can deduce that this form produces $b\binom{a}{2}$ strings. Using a similar reasoning to compute the number of strings produced by other forms with two of the X_i's being the same and combining (3.13) and (3.14) we have that

$$\begin{aligned} M(n,3) \geq |\mathcal{C}| = {} & M(a,3) + M(b,3) + M(c,3) + \\ & + \binom{a}{2}(b+c) + \binom{b}{2}(a+c) + \binom{c}{2}(a+b) + 2abc \end{aligned} \tag{3.15}$$

or equivalently, by rearranging the above formula,

$$\begin{aligned} & \left(M(a+b+c,3) - \binom{a+b+c}{3}\right) \geq \\ & \geq \left(M(a,3) - \binom{a}{3}\right) + \left(M(b,3) - \binom{b}{3}\right) + \left(M(c,3) - \binom{c}{3}\right) + abc. \end{aligned} \tag{3.16}$$

In particular, if n is a power of 3, we can split the underlying set in three equal parts in each step, and the recurrence (3.16) together with the starting value $M(3,3) = 3$ yields

$$M(n,3) \geq \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n, \tag{3.17}$$

proving the second part of our theorem.

Note that the quantity in (3.17) equals $\binom{n}{3} + \frac{1}{24}n^3 + \frac{7}{24}n$. To estimate how far this is from $M(n,3)$ for general n, we seek an upper bound on the remainder term $r(n)$ defined by

$$M(n,3) = \binom{n}{3} + \frac{1}{24}n^3 - r(n).$$

Substituting in (3.16), we obtain

$$\frac{(a+b+c)^3}{24} - r(a+b+c) \geq \frac{a^3+b^3+c^3}{24} + abc - r(a) - r(b) - r(c).$$

Letting a, b and c be appropriate numbers with pairwise differences not exceeding 1, we have

$$\begin{aligned} r(3n) &\leq 3r(n) \\ r(3n-1) &\leq 2r(n) + r(n+1) + \frac{1}{4}n \\ r(3n+1) &\leq 2r(n) + r(n-1) + \frac{1}{4}n. \end{aligned}$$

It easily follows by induction that, for some constant C,

$$r(n) \leq Cn \log n$$

and this proves the lower bound claimed. □

3.5.2 An upper bound on reverse–free strings

Theorem 3.9

$$M(n,3) \leq \begin{cases} \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n & \text{for } n \text{ odd} \\ \\ \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{1}{2}n & \text{for } n \text{ even.} \end{cases}$$

Proof. We begin by introducing some definitions, in order to simplify the forthcoming proof.

Definition 3.10 *Given a string* $\mathbf{x} = x_1x_2x_3$ *we say that its support is* $\{x_1, x_2, x_3\}$, *i.e the set of its three elements.*

Definition 3.11 *Given a set* $\mathcal{C}$ *of strings from* $[n]$, *we say that a support* $\{u, v, w\}$ *has* i *occurrences in* $\mathcal{C}$ *if* $\mathcal{C}$ *contains* i *strings of support* $\{u, v, w\}$.

Now, let $\mathcal{C}$ be a set of pairwise reverse–free strings of elements of $[n]$. Denote by $T_i(\mathcal{C})$ the set of supports having *exactly* i occurrences in $\mathcal{C}$. When the context is clear we will simply write T_i instead of $T_i(\mathcal{C})$. The following claim is at the base of our proof.

Claim 3.12 *Let* $\mathcal{C}$ *be a set of pairwise reverse–free strings from* $[n]$. *Then the following inequalities hold:*

(*i*) $|T_0| + |T_1| + |T_2| + |T_3| = \binom{n}{3}$

(*ii*) $|T_2| + |T_3| \leq \begin{cases} \dfrac{n^3 - n}{24} & \text{for } n \text{ odd} \\ \dfrac{n^3 - 4n}{24} & \text{for } n \text{ even} \end{cases}$

(*iii*) $|T_3| - |T_0| \leq \dfrac{n}{3}$.

Proof (**i**). Recalling that two strings of the same support are reverse–free if and only if one is a cyclic shift of the other we can deduce that any support has at most three occurrences in $\mathcal{C}$ (the set of all cyclic shifts). Hence, $|T_i| = 0$ for every $i > 3$. At this point the first equality of the claim follows immediately by noting that the sets T_i (with $0 \leq i \leq 3$) form a partition of the whole set of the supports.

Given the set $\mathcal{C}$, we create a directed graph G_1 on the vertex set $[n]$ by putting $uv \in E(G_1)$ whenever there exists a vertex x such that $uvx \in \mathcal{C}$. Note that $uv \in$

$E(G_1)$ implies $vu \notin E(G_1)$. If the resulting graph is not a tournament, we add edges arbitrarily to turn it into one. Similarly, let G_2 be a tournament that contains all edges uv such that, for some vertex x, we have $xuv \in \mathcal{C}$, and G_3 a tournament containing all edges uv such that $vxu \in \mathcal{C}$ for some x.

Now define two additional directed graphs, D and M by putting

$$E(M) = \{uv : uv \in E(G_i) \cap E(G_j) \text{ for some pair } i, j \in \{1, 2, 3\}\}$$
$$E(D) = \{uv : uv \in E(G_1) \cap E(G_2) \cap E(G_3)\}.$$

For a directed graph G, let $c(G)$ be the number of cyclically oriented triangles in G. Let $d^+(v)$ and $d^-(v)$ be the out-degree and in-degree of v respectively. It is well known (see for example [36]) that, for a graph G on n vertices,

$$c(G) \leq \binom{n}{3} - \sum_{v \in V(G)} \binom{d^+(v)}{2} \leq \begin{cases} \dfrac{n^3 - n}{24} & \text{for } n \text{ odd} \\ \dfrac{n^3 - 4n}{24} & \text{for } n \text{ even.} \end{cases} \tag{3.18}$$

Proof (**ii**). If uvw and vwu both belong to $\mathcal{C}$, then $uv \in E(G_1)$ and $uv \in E(G_3)$, so $uv \in E(M)$. Similarly, $vw, wu \in E(M)$. We have therefore

$$|T_2| + |T_3| = |T_2 \cup T_3| \leq c(M)$$

and the result follows.

Proof (**iii**). For any vertex u of the graph D define the sets of supports

$$S_u = \{\{u, v, w\} : uv \in E(D) \text{ and } uw \in E(D)\}$$

It is easy to see that $\bigcup_{u \in [n]} S_u \subseteq T_0$. The set S_u consists of all three-element sets containing u and two of its out-neighbors in D, hence $|S_u| = \binom{d^+(u)}{2}$. For any two different vertices u, v the sets S_u and S_v are disjoint, so $|T_0| \geq \sum \binom{d^+(u)}{2}$. Similarly, $|T_0| \geq \sum \binom{d^-(u)}{2}$ holds too. It follows that

$$2|T_0| \geq \sum_{u \in [n]} \left[\binom{d^+(u)}{2} + \binom{d^-(u)}{2} \right]. \tag{3.19}$$

Each string from T_3 induces a cyclic triangle in D. A vertex u in $D(\mathcal{C})$ is in at most $d^+(u) \cdot d^-(u)$ cyclic triangles and hence

$$|T_3| \le c(D) \le \frac{1}{3} \sum_{u \in [n]} d^+(u) \cdot d^-(u). \tag{3.20}$$

Multiply (3.19) by $-1/2$ and add to (3.20). Use that for non-negative integers x, y we have $\frac{1}{3}xy - \frac{1}{4}(x^2 - x) - \frac{1}{4}(y^2 - y) \le \frac{1}{3}$. We obtain

$$|T_3| - |T_0| \le \sum_{u \in [n]} \left[\frac{1}{3} d^+(u) \cdot d^-(u) - \frac{1}{2} \binom{d^+(u)}{2} - \frac{1}{2} \binom{d^-(u)}{2} \right] \le \frac{n}{3}$$

This concludes the proof of the claim. □

Finally, we finish the proof of the Theorem by adding the three inequalities of Claim 3.12. We have

$$\begin{aligned} |\mathcal{C}| &= |T_1| + 2|T_2| + 3|T_3| \\ &= \binom{n}{3} + (|T_2| + |T_3|) + (|T_3| - |T_0|) \\ &\le \begin{cases} \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n & \text{for } n \text{ odd} \\ \\ \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{1}{2}n & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

as claimed. □

3.6 A better upper bound on $M(n, k)$

In this section we will see how to exploit the exact result for $M(n, 3)$ in order to improve the upper bound claimed in Theorem 3.5.

Theorem 3.13

$$M(n, k) \le \left(\frac{5}{24} \right)^{\frac{k}{3}} n^k$$

Proof. Let $\mathcal{C}$ be a maximum reverse–free set of k–strings from the set $[n]$ with $k \geq n$. For simplicity suppose first k is a multiple of 3. Consider a k–string $\mathbf{x} = x_1 \ldots x_k$ in $\mathcal{C}$. Now consider the set S_1 of the projections of all the strings of $\mathcal{C}$ into the first three coordinates $\{1, 2, 3\}$. Obviously, S_i forms a reverse–free set and thus $|S_1| \leq M(n, 3)$. So, there are at most $M(n, 3)$ ways to choose the first three symbols for each of the strings in $\mathcal{C}$. Now, what can we say for the second three elements $x_4x_5x_6$. It is clear that once we fix the first three symbols of the k–string, it remains at most $M(n-3, 3)$ ways to choose the second three elements. This is because repetition of symbols is not permitted in a k–string. Following this reasoning we have that

$$M(n,k) \leq M(n,3)M(n-3,3)\ldots M(n-3(k/3),3) \leq \Big[M(n,3)\Big]^{\lfloor \frac{k}{3} \rfloor}$$

It is not difficult to see that the last inequality is tight in an asymptotic sense. Now, by the previous result we have

$$M(n,k) \leq \left(\frac{5}{24}\right)^{\frac{k}{3}} n^k$$

and this concludes the proof. □

Observe that by Theorem 3.5 we have that $f(k) \leq \dfrac{1}{2^{\lfloor k/2 \rfloor}} \leq 0.708^k$. Now, in the previous theorem we slightly improve the bound in $f(k) \leq 0.59^k$.

Chapter 4

2–Cancellative set families

This chapter is based on the paper [40]

This chapter will discuss problems that impose requirements on subsets of a set of strings. We have already seen such type of problems when introducing hypergraph capacity. Here we will consider in a sense, a generalized version of the capacity problem for the complete k–uniform hypergraph. We will see how this approach leads to a collection of new challenging problems that establish many interesting connections between combinatorics and information theory. Then we will focus on instances of these problems concerning the determination of the maximum cardinality of sets of binary strings of equal length, with requirements that involve triples or four–tuples of strings. In particular, in this context we will review the concept known in literature as cancellative set family [18], which in turn is, as we will see, a special case of the combinatorial version of the zero error transmission through a binary multiplicative channel [60]. Finally, we introduce the concept of a 2-cancellative set family, that we claim is a natural generalization of cancellative families. We provide bounds on the size of 2-cancellative set families without being able to close the gap between them.

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4.1 Introduction

As already mentioned in Chapter 1, the concept of G–difference can be generalized from binary relations to relations that involve larger subsets of strings. As we have seen, problems of uniform hypergraph capacity stem from this extension. These problems are difficult to solve and progress is accordingly slow. Moreover, let us stress that apart from their intrinsic interest, these are connected to many other, seemingly unrelated, problems in information theory, just to mention the problems of perfect hashing [19] and zero error list decoding [15].

Recall that in the capacity problem for k–uniform hypergraphs we ask for the maximum number of n–length strings over an alphabet of m symbols, such that for every k-tuple of them, there exists a coordinate i on which the projections of the strings form an hyperedge. In the case of the complete k–uniform hypergraph this is equivalent to require the existence of a coordinate in which all the k strings differ, i.e. assume different values. This requirement can be considered from a more general point of view: we can interpret this sort of simultaneous distinguishability as requiring the k strings to differ in a small set of coordinates (in the above case in just one coordinate). On the base of this consideration the authors of [46] introduced the following natural generalization of the capacity problem regarding the complete k–uniform hypergraph:

Problem: Given a set of strings of length n over an alphabet of m symbols, we require that for any r strings in the set, there exists a set of coordinates of cardinality $q(r, m)$ on which the projections of the strings are all different.

The authors of [46] began the study of these issues by considering the above problem in the case of binary strings, i.e. strings over the alphabet $\{0, 1\}$. Observe that if we consider an n–length binary string as the characteristic vector of a subset of a set of n elements, problems for the binary case can be reformulated in an extremal set theory language. The decision to focus on this case, is also fueled by this consideration.

The first non trivial problem they considered is the case when $r = 4$ and $q(r, 2) = 2$. Precisely, in this case we ask for the largest set of binary strings such that for any four of them there exist two different coordinates i, j in which the strings are all different, to say it better the projections of the strings onto these two coordinates are all the four possible binary strings of length 2. In order to simplify the forthcoming discussion let us first introduce some definitions. First we will refer to the ordered triple (w_i, x_i, y_i, z_i) as the i'th column of the ordered four–tuple of binary strings $(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z})$. Furthermore the weight of a binary string $\mathbf{x}$ of length n is given by $w(\mathbf{x}) = \sum_{i=0}^{n} x_i$, i.e. the number of 1's in the coordinates of $\mathbf{x}$. Now, it is not difficult to see that the requirement of the above problem can be reformulated in an equivalent way as follows: for every four distinct strings in the set $\mathbf{w}, \mathbf{x}, \mathbf{y}$ and $\mathbf{z}$, considered in an arbitrary but fixed order $(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ there exist two coordinates i and j for which the corresponding columns are different, not complementary and of weight 2. This problem, introduced in [46], served as a starting point to many fruitful investigations. Indeed, it is not difficult to imagine how to pose new problems in the same spirit as this one. Mostly, this leads to interesting and difficult instances. Just to begin, consider the simplest modified versions of the above problem that one can think of: for any four binary strings we impose the existence of just one column of weight 2 or in another case we may require the existence of all the three possible columns of weight 2 no two of which are complementary. Both these problems are difficult and still unsolved (see for example [3, 21, 31]).

In this chapter we focus on problems that involve requirements concerning the existence of t columns of weight 1 (for some integer t). We will first require the above property to hold for every three binary strings in the set and then we extend and ask the property to be satisfied by any four strings. In particular we will restrict attention to a special instance of the latter case. We have already mentioned that these problems may also have an interesting reformulation in an extremal set theory context. Moreover, we will see that many of them are also reformulations of instances arising in an information theory context. This is an interesting feature because as it is often the case, when questions are posed across the disciplines, the results obtained have a bearing on each other. Now, obviously not all of the problems involving requirements on sets of strings have both of these reformulations. In many cases they are not currently interpreted in an information theory context. However, it is necessary to note that even when this is not the case the study of a formally information–theoretic problem may reveal important connections between the solutions of similar problems and thus lead to a better understanding of the topic as a whole. Moreover, if a problem seems to be of an information theory flavor, because a similar problem can be reformulated in information theory, then it is reasonable to assume that sooner or later it will be applicable to a practical instance not yet emerged. Thus, we think that the study of these type of problems reveals a deep connection between information theory and combinatorics, as well as the need to combine tools from both areas in order to progress.

4.1.1 Requirements regarding triples of strings

Here we briefly present some problems that involve requirements on triples of strings. To this purpose consider the following:

Problem: We ask for the maximum number of binary strings of length n, such that for any 3 of them there exist at least r coordinates $i_1, \ldots, i_r$ for which the corresponding columns are all different and of weight 1 (obviously $r \in [3]$).

It is interesting to note how this simple looking problem is directly connected to many intriguing, well known and unsolved problems of both combinatorics and information theory, most of which have been studied extensively in the literature. For instance, when $r = 1$, the above problem corresponds to the famous problem on strong Δ–systems (also known as sunflowers) [49] about which very few is known. In view of this consideration it is somehow unexpected that for the case $r = 2$ the asymptotic behavior is completely determined. This case is connected to perhaps one of the most famous problems in extremal set theory regarding cancellative set families, originally introduced by Erdős and Katona [32].

Definition 4.1 *A family $\mathcal{F}$ of subsets of the set $[n]$ is called cancellative if for all A, B, C in $\mathcal{F}$, $A \cup B = A \cup C$ implies $B = C$.*

As always, we assume without loss of generality that the n element ground set is $[n]$. We are interested in the maximum size of a cancellative family of subsets of $[n]$. Now, if we represent each set by its characteristic vector of length n, it becomes clear that this corresponds to our previous problem in which $r = 2$. However, let us explicitly restate the definition of a cancellative set of strings.

Definition 4.2 *A set F of binary strings of length n is called cancellative if for any three strings $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in the set, there exist two coordinates i and j for which the ordered triples (x_i, y_i, z_i) and (x_j, y_j, z_j) are distinct and of weight 1.*

Now, much has been written about the cancellative problem. In [18], Frankl and Füredi have introduced an upper bound for the maximum cardinality of cancellative set families as a function of n. This problem remained opened for a long time, until Shearer [58] disproved the corresponding conjecture of Erdős and Katona [32] and this led the way to Tolhuizen's beautiful discovery [60] that the Frankl and Füredi upper bound is tight in the sense of exponential asymptotics. It is interesting to note that the solution came from information theory. Indeed, Tolhuizen considered the following scenario: Two terminals, T_1 and T_2, wish to send messages to a common receiver over a binary multiplying channel. To this end, they choose the input sets F_1 and F_2, respectively, of binary strings of length n. Now the channel is multiplying, which means that if $\mathbf{x} \in F_1$ and $\mathbf{y} \in F_2$ are fed to the channel, it gives as output the string $\mathbf{x} \cdot \mathbf{y}$, defined by $(\mathbf{x} \cdot \mathbf{y})_i = x_i \cdot y_i$ for every coordinate $i \in [n]$. Each terminal should be able to determine unambiguously the string transmitted by the other one, using its own transmitted string and the observed channel output. Now, a pair of sets (F_1, F_2) that satisfy this requirement is called "a uniquely decodable" code, or simply an UD code. Moreover we restrict to the case when $F_1 = F_2 = F$. In this case we talk about symmetric UD code. It is easy to see that such a code is strongly connected to the cancellative set problem. To this purpose it suffices to observe that a cancellative set of binary strings corresponds to a symmetric UD code. Indeed, if F is a cancellative set then by definition one can observe that for any three of its elements, $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ we have that $\mathbf{x} \otimes \mathbf{y} \neq \mathbf{x} \otimes \mathbf{z}$ (where $\otimes$ is the binary addition) which implies $\bar{\mathbf{x}} \cdot \bar{\mathbf{y}} \neq \bar{\mathbf{x}} \cdot \bar{\mathbf{z}}$ (where $\bar{\mathbf{x}}$ is the complement of the binary string $\mathbf{x}$). Hence, F is a symmetric UD code. In [60] Tolhuizen showed that it is possible to construct symmetric UD codes of cardinality that asymptoticly achieve the upper bound of [18], determining in this way the exact asympotitc of maximum cardinality of a cancellative set.

We have stressed many times that although most of the problems we have presented so far, have been studied systematically, almost none of them has been solved even in an asymptotic sense. Tolhuizen's solution regarding the maximum size of cancellative set families is almost the only notable exception (for another example see [7, 23, 35]). Given the exceptional status of this problem among similar ones about families with excluded triples of subsets, inasmuch as no other problem of a similar kind has been solved, it is reasonable to look for an analogous property concerning excluded four-tuples, in the hope that this might give rise to the "easiest" of all such problems for four-tuples. Moreover relations between four strings are expressible through requirements regarding two pairs of strings and thus easily transformed into graph covering problems (we can consider graphs whose vertices are pairs of strings and where arcs represent the constraints between pairs) and in these cases sometimes technical tools from information theory as entropy, etc. are used with great success. In the next section we define what we claim is a natural generalization of the cancellative property. The corresponding problem has not been treated before in the literature, even though corresponding bounds can be derived from those for similar properties. In particular, we will improve on the upper bound obtainable from the one in [17] for a similar but weaker property.

4.1.2 Requirements regarding four–tuples of strings

Here we will discuss issues involving relations of four strings. The following problem is a natural generalization of the one presented in the previous subsection:

Problem: We ask for the maximum number of binary strings of length n, such that for any 4 of them there exist at least r coordinates $i_1, \dots, i_r$ for which the corresponding columns are all different and have weight 1 (obviously $r \in [4]$).

Again this simple problem has many connections with well known ones. To see this we briefly recall some of these straightford connections:

The case $r = 1$: This case corresponds to the problem of so called 4–*locally thin sets* originally introduced by Alon *et al.* [2]. Let $\mathcal{F}$ be a family of subsets of $[n]$ elements. We say that the family is 4–locally thin if for any 4 of its distinct member sets at least one point $i \in [n]$ is contained in exactly one of them. We are interested in finding out how large these families can get. Now, this problem is still unsolved and the best upper bound comes from [17]. It is worth to mention that this problem can be formulated in a more general way and thus one can consider k–locally thin sets. It is clear that this presents particular difficulties. It suffices to mention that the case of 3–locally thins sets corresponds to the problem on strong Δ–systems (also known as sunflowers) [49] already mentioned in the previous subsection.

The case $r = 4$: In this case we get back a well-known problem, the one about the largest cardinality of a 3-*cover–free* family (see, for example [20], [30]). A family of sets $\mathcal{F}$ is called 3–cover–free if for any four distinct elements of $\mathcal{F}$, A_0, A_1, A_2 and A_3 we have that $A_0 \not\subseteq A_1 \cup A_2 \cup A_3$. It is worth to note that this problem was introduced by Kautz and Singleton [34] in 1964 concerning binary codes. Moreover their results were rediscovered several times in information theory, in combinatorics [16], and in group testing [28].

Unfortunately nothing more is known for the case $r = 2$ or $r = 3$. The best upper bound known for both these problems remains the bound on 4-locally thin sets [17]. However, we will improve this bound in the case $r = 3$. In the rest of this chapter we focus on this problem.

4.2 2–cancellative set families

In this section we will study 2–cancellative set families and provide bounds regarding their maximum cardinality.

Definition 4.3 *A family $\mathcal{F}$ of subsets of an n-set is called* 2-cancellative *if or any four of its sets A, B, C, D we have $A \cup B \cup C = A \cup B \cup D$ implies $C = D$.*

One can immediately see that representing each set by its characteristic vector of length n, we can define a 2–cancellative set of strings.

Definition 4.4 *A set F of binary strings of length n is called 2–cancellative is for every four-tuple $\{\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ of distinct strings in it, considered in an arbitrary but fixed order $(\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ there exist at least three different values of $r \in [n]$, such that the corresponding k'th columns (w_r, x_r, y_r, z_r) are all different and of weight 1.*

Let $M(n)$ be the maximum size of such a set of strings. We would like to determine the exponential asymptotics

$$t(4) = \limsup_{n \to +\infty} \frac{1}{n} \log M(n).$$

Now, it follows from the main result of [17], that $t(4) < 0.4561$. The following theorem improves this result by showing that $t(4) < 0.42$.

Theorem 4.5

$$0.11 < t(4) \leq 0.42$$

Proof: In order to prove the lower bound, we first reformulate the 2-cancellative property. Consider the ordered four-tuple of different binary strings $(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ of length n. We require that the following holds

$$\exists i \in [n] \quad \text{such that} \quad (\{w_i, x_i\}, \{y_i, z_i\}) = (\{0, 0\}, \{0, 1\}). \tag{4.1}$$

For the sake of brevity we say that the underlying set $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ has the *critical* property if the corresponding relation (4.1) holds for any ordering of its strings. Observe that once the set of four distinct strings is fixed, the configuration of ordered pairs of unordered couples that we introduced is uniquely determined by the first couple of the pair. Therefore, given an arbitrary set $T = \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ of distinct binary strings, we will refer to each of the configurations it generates, by the the first couple $A \in \binom{T}{2}$ of the ordered pair. It can be seen that T has the critical property if and only if it is 2-cancellative. The "if" part is obvious. If we have three different columns of length 4 and weight 1, then, necessarily, for every couple $A \in \binom{T}{2}$, one of these strings has a 0 in both positions defined by the strings of A. On the other hand, if we have a set of columns such that for every A at least one of them satisfies (4.1), then they must contain at least three different columns of length 4 and weight 1; in fact, if there were only two of them, then their positions of 1's would define a couple A for which the relation (4.1) is not satisfied. Thus, we can say that a set F of binary strings is 2-cancellative if every four-tuple of its strings $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ has the critical property.

The lower bound is obtained by a standard random choice argument. Let F be a random collection of N binary strings of length n, constructed by letting each coordinate of each member of F be, randomly and independently 1 with probability p and 0 with probability $1-p$ for a p that will be chosen later. Now, the expected number of configurations in F which don't have the critical property is $\binom{N}{4}\binom{4}{2}[1-2p(1-p)^3]^n$. By deleting an arbitrarily chosen string from each of these forbidden configurations we obtain a set F of strings having the 2-cancellative property, of cardinality

$$|F| \geq N - 6\binom{N}{4}[1-2p(1-p)^3]^n.$$

Choosing $p = 1/4$, $N = \lfloor (101/128)^{n/3} \rfloor$ and recalling that $M(n) \geq |F|$, it follows that

$$t(4) = \limsup_{n\to+\infty} \frac{1}{n} \log M(n) \geq \frac{1}{3}(7 - \log 101) > 0.11$$

as claimed.

We next give the proof of the upper bound.

Consider a 2-cancellative set of binary strings of length n, with the additional property that all its members have the same weight. Let $N(n)$ be the maximum cardinality of such a set. It is clear that $N(n)$ and $M(n)$ have the same exponential asymptotics, as we have that $N(n) \leq M(n) \leq (n+1)N(n)$. As we are only interested in the asymptotic behavior of $M(n)$, we can restrict ourselves to the case of sets in which each of the member strings has the same weight. Let $F = F_n$ be such a set, which achieves maximum cardinality. In other words, the strings in F have the same weight equal to np for some p, $0 \leq p \leq 1$. In order to proceed with the proof we next give some definitions:

Given a string $\mathbf{x}$, its *projection* onto the set of coordinates $I = \{i_1, ...i_m\}$, (with the

natural ordering $i_1 <, \ldots, < i_m$), is the string $\mathbf{x}|_I = x_{i_1}...x_{i_m}$.
For every $\mathbf{x} \in F$ define the set of all the coordinates where $\mathbf{x}$ has a zero

$$I_{\mathbf{x}} = \{i : i \in [n]; x_i = 0\}.$$

Obviously, $|I_{\mathbf{x}}| = (1-p)n$.
For all $\mathbf{x} \in F$ define the projection of F onto the set of coordinates $I_{\mathbf{x}}$ as follows:

$$F^{\mathbf{x}} = \{\mathbf{y} : \mathbf{y} \in \{0,1\}^{(1-p)n}; \exists \mathbf{z} \in F \setminus \{\mathbf{x}\}; \mathbf{y} = \mathbf{z}|_{I_{\mathbf{x}}}\}.$$

Now let us fix $\mathbf{x} \in F$ and consider $F^{\mathbf{x}}$. The next observations follow directly from the 2-cancellative property:

1. $|F| = |F^{\mathbf{x}}| + 1$.

2. For every three strings $\mathbf{w}, \mathbf{y}, \mathbf{z}$ in the set $F^{\mathbf{x}}$ there exist at least two integer values $k \in [0, n(1-p)]$, such that the ordered triples (w_k, y_k, z_k) are all different and $w_k + y_k + z_k = 1$.

Indeed, let $(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ be an arbitrary four-tuple of distinct strings in the set F, where $\mathbf{x}$ is the fixed element of F. We know that F is a 2-cancellative family. Hence there exist at least three different values of k, such that the ordered quadruples (w_k, x_k, y_k, z_k) are all different and $w_k + x_k + y_k + z_k = 1$. This implies that $x_k = 0$ for at least two different values of $k \in [n]$; observe that this forces k to belong to $I_{\mathbf{x}}$. Furthermore, in correspondence to these values of k we have at least two different triples (w_k, y_k, z_k) such that $w_k + y_k + z_k = 1$. Now, recalling that $k \in I_{\mathbf{x}}$ and considering the projections $w|_{I_{\mathbf{x}}}, y|_{I_{\mathbf{x}}}, z|_{I_{\mathbf{x}}}$, it is easy to verify the properties we claimed.
We can view $F^{\mathbf{x}}$ as the set of characteristic strings of a family $\mathcal{F}^{\mathbf{x}}$ of subsets of an $n(1-p)$-set. In the terminology of Frankl and Füredi [18], $\mathcal{F}^{\mathbf{x}}$ is a *cancellative* family. In [18], they estimated the size of the largest cancellative family $\mathcal{F}_k$, consisting of k-element subsets of $[n]$. They proved that if $k \leq n/2$ then $|\mathcal{F}_k| \leq \binom{n}{k} 2^k / \binom{2k}{k}$.

Thus,

$$|F^{\mathbf{x}}| = |\mathcal{F}^{\mathbf{x}}| \leq \sum_{k=0}^{\hat{k}} \binom{\hat{n}}{k} 2^k / \binom{2k}{k}, \tag{4.2}$$

where $\hat{n} = n(1-p)$ is the length of the binary strings of $F^{\mathbf{x}}$ and $\hat{k}$ is their maximum weight. The following claim is at the base of our proof.

Claim 4.6 *Let F be the set of characteristic strings of a 2-cancellative family $\mathcal{F}$ of subsets of cardinality np, (for some $0 < p \leq 1$), of an n-set. Let ε be an arbitrary nonnegative constant, to be specified later. There is a binary string $\mathbf{x} \in F$ and a constant $\gamma = \gamma_{(\varepsilon,p)}$, $(0 < \gamma_{(\varepsilon,p)} \leq 1)$, such that at least $\gamma|F^{\boldsymbol{x}}| - 1$ strings in $F^{\boldsymbol{x}}$ have at most a weight $n(1-p)(p+\varepsilon)$.*

Proof: For each coordinate j, $(1 \leq j \leq n)$, denote by $c_j = \sum_{\mathbf{y} \in F} y_j$, the sum of the projections of all the strings of F, to the j'th coordinate. With this notation, the overall weight of the projections of the strings of F onto every possible set of coordinates $I_{\mathbf{x}}$, $(\mathbf{x} \in F)$, can be expressed as

$$\sum_{\mathbf{x} \in F} \sum_{\mathbf{y} \in F^{\mathbf{x}}} w(\mathbf{y}) = \sum_{\mathbf{x} \in F} \sum_{j \in I_{\mathbf{x}}} c_j = \sum_{j=1}^{n} c_j(|F| - c_j). \tag{4.3}$$

The first equality is simply obtained by "double counting" the number of coordinates equal to one in the set F (observe that x_j is also considered when calculating c_j, but its contribution to the sum is zero, as $x_j = 0$ when $j \in I_{\mathbf{x}}$). The second equality follows by observing that each c_j appears in the precedent sum in correspondence with all the strings $\mathbf{x} \in F$ for which $j \in I_{\mathbf{x}}$, in other words, for all the strings $\mathbf{x}$ such that $x_j = 0$. Obviously, the number of strings having zero at their j'th coordinate is $|F| - c_j$. Now, by a simple application of Jensen's inequality to the function x^2, we can upper bound the right-most end of (4.3) by

$$\sum_{j=1}^{n} c_j|F| - \sum_{j=1}^{n} {c_j}^2 \leq \sum_{j=1}^{n} c_j|F| - \frac{1}{n}\Big(\sum_{j=1}^{n} c_j\Big)^2. \tag{4.4}$$

Again, a simple "double counting" argument shows that

$$\sum_{j=1}^{n} c_j = \sum_{\mathbf{y} \in F} w(\mathbf{y}) = |F|pn,$$

whence we can rewrite the last expression of (4.4) as

$$|F|^2np - \frac{1}{n}|F|^2n^2p^2 = |F|^2n(1-p)p. \tag{4.5}$$

Hence, recalling that $|F| = |F^{\mathbf{x}}| + 1$, the relations (4.3)-(4.5) give the upper bound

$$\frac{1}{|F|}\sum_{\mathbf{x}\in F}\sum_{\mathbf{y}\in F^{\mathbf{x}}} w(\mathbf{y}) \leq (|F^{\mathbf{x}}|+1)n(1-p)p.$$

This immediately shows the existence of an $\mathbf{x} \in F$ such that

$$\sum_{\mathbf{y}\in F^{\mathbf{x}}} w(\mathbf{y}) \leq (|F^{\mathbf{x}}|+1)n(1-p)p. \tag{4.6}$$

Given such an $\mathbf{x}$, we will show that a constant fraction of the strings of $F^{\mathbf{x}}$ has a weight smaller than $(1-p)n(p+\varepsilon)$. Observing that a string $\mathbf{y}$ in $F^{\mathbf{x}}$ has a length equal to $n(1-p)$, we can define its density $p_{\mathbf{y}}, (0 < p_{\mathbf{y}} \leq 1)$ as

$$p_{\mathbf{y}} = \frac{w(\mathbf{y})}{n(1-p)}.$$

Denote by $F_1^{\mathbf{x}}$ the set of all the strings $\mathbf{y}$ of $F^{\mathbf{x}}$ for which we have $p_{\mathbf{y}} < p+\varepsilon$ and by $F_2^{\mathbf{x}}$ its complement in $F^{\mathbf{x}}$, i.e., $\mathbf{y} \in F_2^{\mathbf{x}}$ whenever $p_{\mathbf{y}} \geq p+\varepsilon$. Thus,

$$\sum_{\mathbf{y}\in F^{\mathbf{x}}} w(\mathbf{y}) \geq |F_2^{\mathbf{x}}|n(1-p)(p+\varepsilon).$$

Combining this with the inequality (4.6), we get

$$|F_2^{\mathbf{x}}| \leq \frac{p}{p+\varepsilon}(|F^{\mathbf{x}}|+1).$$

Recalling that $|F_2^{\mathbf{x}}| = |F^{\mathbf{x}}| - |F_1^{\mathbf{x}}|$ we obtain

$$|F_1^{\mathbf{x}}| \geq \frac{\varepsilon}{p+\varepsilon}|F^{\mathbf{x}}| - \frac{p}{p+\varepsilon}. \tag{4.7}$$

This bound shows that there is a constant $\gamma = \dfrac{\varepsilon}{p+\varepsilon}$, $(0 < \gamma_{(\varepsilon,p)} \leq 1)$, such that at least $\gamma|F^{\mathbf{x}}| - 1$ strings in $F^{\mathbf{x}}$ have a weight at most $n(1-p)(p+\varepsilon)$. This completes the proof of the claim.

Now we are ready to prove the theorem. Applying the previous result to our set F, for every fixed $\varepsilon > 0$ we can find an $\mathbf{x} \in F$ and a γ, such that at least a γ-fraction of the strings of $F^{\mathbf{x}}$, has a weight smaller than $\widehat{k} = n(1-p)(p+\varepsilon)$. Combining this with the Frankl-Füredi bound [18] we have for $0 < p \leq 1/2$ that

$$\gamma|F^{\mathbf{x}}| - 1 \leq |F_1^{\mathbf{x}}| \leq \sum_{k=0}^{\hat{k}} \binom{\hat{n}}{k} 2^k / \binom{2k}{k}, \tag{4.8}$$

where $\gamma = \dfrac{\varepsilon}{p+\varepsilon}$, and since we are considering strings in $F^{\mathbf{x}}$, $\hat{n} = (1-p)n$, therefore $\hat{k} = \hat{n}(p+\varepsilon) = n(1-p)(p+\varepsilon)$. We need the following exponential bound for binomial coefficients, (see, *e.g*, Section 1.2 in Csiszár and Körner [13])

$$\binom{\hat{n}}{k} \leq \exp_2\left(\hat{n} h\left(\frac{k}{\hat{n}}\right)\right) \quad , \quad \binom{2k}{k} \geq \frac{1}{2k+1} \exp_2(2k) \tag{4.9}$$

Combining (4.8) and (4.9) we have

$$\gamma|F^{\mathbf{x}}| \leq (\hat{k}+1) \max_{k \leq \hat{k}} \binom{\hat{n}}{k} \exp_2(k) / \binom{2k}{k} \tag{4.10}$$

$$\leq \hat{n} \exp_2\left(\hat{n}\left(\max_{k \leq \hat{k}}\left(h\left(\frac{k}{\hat{n}}\right) - \frac{k}{\hat{n}} + \frac{\log \hat{n}}{\hat{n}}\right)\right)\right). \tag{4.11}$$

Thus,

$$\begin{aligned} \frac{1}{n} \log |F^{\mathbf{x}}| \leq \frac{1}{n} \log\left(\frac{n(1-p)}{\gamma}\right) \\ + (1-p) \max_{k \leq \hat{k}}\left(h\left(\frac{k}{\hat{n}}\right) - \frac{k}{\hat{n}} + \frac{\log \hat{n}}{\hat{n}}\right). \end{aligned} \tag{4.12}$$

Now, set $q = \dfrac{k}{\hat{n}}$ and rewrite the right hand side of (4.12) as

$$= \frac{1}{n} \log\left(\frac{n(1-p)}{\gamma}\right) + (1-p) \max_{q \leq p+\varepsilon}\left(h(q) - q + \frac{\log \hat{n}}{\hat{n}}\right). \tag{4.13}$$

Thus,

$$
\begin{aligned}
t(4) &= \limsup_{n\to+\infty} \frac{1}{n}\log M(n) && (4.14)\\
&\le \limsup_{n\to+\infty} \frac{1}{n}\log\big((n+1)N(n)\big) && (4.15)\\
&\le \limsup_{n\to+\infty}\Bigg(\frac{1}{n}\log(n+1)+\frac{1}{n}\log\left(\frac{n(1-p)}{\gamma}\right) \\
&\qquad + (1-p)\max_{q\le p+\varepsilon}\left(h(q)-q+\frac{\log\hat{n}}{\hat{n}}\right)\Bigg). && (4.16)
\end{aligned}
$$

Whence it follows that

$$
t(4) \le \max_{0\le p\le\frac{1}{2}}(1-p)\max_{q\le p+\varepsilon}\Big(h(q)-q\Big). \tag{4.17}
$$

Now, since $h(q)-q$ is monotonically increasing in $[0,1/3]$ and monotonically decreasing elsewhere, we have for every $\varepsilon>0$

$$
\max_{0\le p\le\frac{1}{3}-\varepsilon}(1-p)\max_{q\le p+\varepsilon}\Big(h(q)-q\Big) \le \max_{p\le\frac{1}{3}-\varepsilon}(1-p)\Big(h(p)-p\Big) \tag{4.18}
$$

and

$$
\max_{\frac{1}{3}-\varepsilon< p\le\frac{1}{2}}(1-p)\max_{q\le p+\varepsilon}\Big(h(q)-q\Big) \le \max_{\frac{1}{3}-\varepsilon< p\le\frac{1}{2}}(1-p)\left(h\left(\frac{1}{3}\right)-\frac{1}{3}\right). \tag{4.19}
$$

In conclusion, (4.17), (4.18) and (4.19) can be summarized as

$$
\begin{aligned}
t(4) &\le \max\left\{\max_{p\le\frac{1}{3}-\varepsilon}(1-p)(h(p)-p),\ \max_{\frac{1}{3}-\varepsilon<p\le 1/2}(1-p)\left(h\left(\frac{1}{3}\right)-\frac{1}{3}\right)\right\}\\
&= \max\left\{\max_{p\le\frac{1}{3}-\varepsilon}(1-p)(h(p)-p),\ \max_{\frac{1}{3}-\varepsilon<p\le 1/2}(1-p)(\log 3-1)\right\}.
\end{aligned}
$$

Choosing $\varepsilon=0.01$ we obtain

$$
t(4) \le \max\big\{0.42, 0.4\big\} = 0.42
$$

as claimed. □

Concluding remarks

In this thesis we have discussed several problems arising from the interplay between combinatorics and information theory. The zero error capacity problem introduced by Shannon [57] lead to many generalizations providing a countless number of intriguing (and almost always still unsolved) new questions. Initially, these capacity type problems arose in an information theory context, but then got translated to extremal combinatorics in order to apply techniques and results from this latter area. However, with the evolution of the graph capacity concept, techniques have begun to flow in the other direction as well, with information theory concepts proving to be useful tools for the solution of problems in combinatorics. In this way these two disciplines have entered a strong interaction enriching both of them. As we have seen, the arguments of this thesis confirm this. We investigate generalizations of capacity problems, consider them from different viewpoints, thereby establishing similarities and connections so as to shed light on the topic as a whole. In particular, we would like to repeat our view that in this context natural open problems abound. Throughout our discussion we have formulated many open problems which appear to be difficult.

In Chapter 2 we have explored the problem of permutation capacity of an infinite graph. We have studied G–different permutations for particular classes of infinite graphs G. Except of some particular cases, these problems present many difficulties and remain unsolved. However, we believe that beyond from their intrinsic interest, a deeper understanding of these problems would lead to a progress in both fields of inquiry.

In Chapter 3 we have provided the complete solution for the problem of reverse–free 3–strings. However, the other cases remain unsolved and the actual bounds are far from being tight. It would be nice to completely solve this problem in the general case, i.e. for any fixed integer k. A major challenge is to determine the maximum size of a reverse–free set of permutations, or at least the asymptotic behavior of this absolutely non-trivial question.

As we have already mentioned in Chapter 4, the asymptotic version of the Frankl–Füredi upper bound [18] for cancellative set families is tight, as proved by Tolhuizen [60]. This is almost (see [7, 23, 35]) the only non–trivial problem in extremal set theory where for an excluded configuration of size greater than two the exact exponential asymptotics is known. Unfortunately, in our generalization to the 2–cancellative set families, there remains a huge gap between the two bounds on the exponent.

We hope we were able to convince the reader of the many challenges of this topic, as well as to point out how combinatorics and information theory ideas get entwined in many cases in resplendent ways.

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